POLYNOMIAL SPLITTINGS OF CASSON-GORDON INVARIANTS

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To my parents.

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Abstract

A knot is called *slice* if it is the boundary of a locally flat 2-dimensional disk in the 4-dimensional ball. Two knots are called *concordant* if the connected sum of one and the mirror image of the other is slice. The set of concordance classes of knots forms an abelian group under connected sum, called *the knot concordance group*. Casson-Gordon invariants detect whether a knot is slice or not. There is a polynomial associated to a knot, called the Alexander polynomial.

In the first half of this thesis we prove that Casson-Gordon invariants of the connected sum of two knots split when the Alexander polynomials of the knots are relatively prime. An important source of examples in knot theory comes from the families of knots called *twisted doubles*. It was unknown if these families are linearly independent in the knot concordance group. As an application of the splitting of Casson-Gordon invariants, we show that all but finitely many twisted doubles in each family are linearly independent in the knot concordance group.

In the second half of this thesis we investigate double concordance classes. A knot is called *doubly slice* if it is the slice of some smooth unknotted 2-sphere in the 4-sphere. Corresponding to double slice knots there is a group, the *double concordance group*. It has been known that every double concordance class contains a prime knot. We give a new and simple proof of this.

Contents

CHAPTER 1

Introduction

A knot is called *slice* if it is the boundary of a locally flat 2-dimensional disk in the 4-dimensional ball. Two knots are called *concordant* if the connected sum of one and the mirror image of the other is slice. The set of concordance classes of knots forms an abelian group under connected sum, called the *knot concordance group*, $\mathcal{C}$. We denote the concordance class of a knot K by K again for simplicity.

In his classification of the knot concordance group, Levine [**Lev2**] defined the algebraic concordance group, $\mathcal{G}$, of Witt classes of Seifert matrices and a homomorphism from the concordance group $\mathcal{C}$ of knots in the 3-sphere S^3 to $\mathcal{G}$. It follows from Levine's work [**Lev1**] that if the connected sum of two knots with relatively prime Alexander polynomials maps to zero in $\mathcal{G}$, then so does each knot.

Casson and Gordon [**CaG**] proved that the kernel of Levine's homomorphism $\mathcal{C} \to \mathcal{G}$ is nontrivial. Gilmer [**G2**] combined the slicing obstructions of Levine with those of Casson and Gordon in a nontrivial way. Using this he defined a Witt type group Γ^+ and showed that there are homomorphisms $\mathcal{C} \to \Gamma^+ \to \mathcal{G}$. Roughly speaking, the group Γ^+ is characterized by the following property: A class of knots maps to zero in Γ^+ if and only if the Casson-Gordon invariants of a knot in the class vanish in Gilmer's sense. Gilmer [**G1**] provided examples to show vanishing in Γ^+ is stronger than being algebraic slice and having Casson-Gordon's obstructions vanish.

In Chapter 2 we show a similar result to the polynomial splitting of the slicing obstructions of Levine in Casson-Gordon invariants as follows.

THEOREM 1.1. *Let K_1 and K_2 be knots with relatively prime Alexander polynomials in $\mathbb{Q}[t^{\pm 1}]$. Suppose that either K_1 or K_2 has a non-singular Seifert form. Then if $K_1 \# K_2$ is zero in Γ^+ then so are both K_1 and K_2.*

To demonstrate the strength of this result we apply it in two settings. The first is to study the family of n-twisted doubles of a given knot K, denoted $D_n(K)$. Casson and Gordon [**CaG**] found the first examples of nontrivial concordance classes in the kernel of Levine's homomorphism using the family $D_n(U)$ where U is the unknot. Since then Jiang [**J**], Litherland [**Lit2**], Livingston and Naik [**LN1, LN2**], and Tamulis [**Tam**] have found infinite linearly independent families in the kernel of Levine's homomorphism among the knots $D_n(K)$. In each case these families were very scarce: roughly one knot was chosen for each prime or prime power integer. Theorem 1.1 will quickly yield as a corollary that for every knot K the family $D_n(K)$ is (with finite exceptions) linearly independent.

THEOREM 1.2. *(a) All but finitely many twisted doubles of a knot are linearly independent in Γ^+ and so in $\mathcal{C}$. More precisely, for any knot K, the set $\mathcal{K} = \{n \in \mathbb{Z} \mid D_n(K)$ has infinite order in $\Gamma^+\}$ has finite complement in $\mathbb{Z}$. For any distinct i and j in $\mathcal{K}$, the classes $D_i(K)$ and $D_j(K)$ are distinct in Γ^+ and the set $\{D_n(K)\}_{n\in\mathcal{K}}$ is linearly independent in the $\mathbb{Z}$-module Γ^+.*

(b) If $\sigma_r(K) \geq 0$ for all r, then $\mathcal{K}$ contains $\mathbb{Z} - \{0, 1, 2\}$, where σ_r denote the Tristram-Levine signature. Moreover, if $\sigma_r(K) > 0$ for $r = \frac{2}{9}, \frac{1}{3}, \frac{2}{5}$, then $\mathcal{K}$ is $\mathbb{Z} - \{0\}$.

We remark that for any knot K, $D_0(K)$ is topologically slice since its Alexander polynomial is a unit in $\mathbb{Z}[t^{\pm 1}]$ and any knot with Alexander polynomial 1 is topologically slice [**FQ**]. Hence the index set $\mathcal{K}$ never contains 0.

COROLLARY 1.3. *(a) For the unknot U, the set $\{D_n(U)\}_{n\in\mathbb{Z}-\{0,1,2\}}$ is linearly independent in Γ^+.*

(b) There are infinitely many knots K for which the set $\{D_n(K)\}_{n\in\mathbb{Z}-\{0\}}$ is linearly independent in Γ^+.

As a second application of Theorem 1.1 we construct examples of linearly independent algebraically slice knots with homology groups of the same order on all prime power fold branched covers. In the past construction of independent knots depended on finding knots for which some branched covers had homology groups of order divisible by distinct primes. Such an approach could conceivably work with doubled knots by using high degree covers, but the argument would be far more burdensome than the one we give. In the case of this second set of examples no such approach could possibly work, and our approach using the splitting associated to the polynomial is definitely required.

In Chapter 3 we investigate double concordance classes. A knot is called *doubly slice* if it is the slice of some smooth unknotted 2-sphere in the 4-sphere. Corresponding to double slice knots there is a group called the *double concordance group*, studied in [**Lev3, Rub1, Sto2**].

Kirby and Lickorish [**KL**] showed that every knot in S^3 is concordant to a prime knot, equivalently, every concordance class contains a prime knot. Generalizations appear in [**Liv, My1, My2, Som**]. Sumners [**Sum**] introduced the notion of invertible concordance. It was proved by Nakanishi [**Nak**, Theorem 3.5] that the Kirby and Lickorish's result can be strengthened:

THEOREM 1.4. *Every knot in S^3 is invertibly concordant to a prime knot.*

We give a new and simple proof of this theorem. A consequence of this is that every double concordance class contains a prime knot.

CHAPTER 2

Splittings of Casson-Gordon invariants

1. Background

In this section we will summarize basic results from classical knot theory. A reference is Rolfsen [**Rol**]. We will work in the category of locally flat oriented manifolds and pairs throughout this chapter; the result all apply in the smooth category.

1.1. Definition of a knot. A *knot* is an oriented homeomorphism class of an oriented pair (S, K), where S is homeomorphic to the 3-sphere S^3 and K is homeomorphic to the circle S^1. We will sometimes abbreviate a class (S, K) by simply K and assume $S = S^3$.

1.2. Knot groups and cyclic covers. It is well-known that the fundamental group of the complement of a knot K in S^3, $S^3 - K$, has a Wirtinger presentation in which all generators are of infinite order and conjugate to each other, and hence its abelianization is infinite cyclic $\mathbb{Z}$. Thus there is a surjective homomorphism ρ from $\pi_1(S^3 - K)$ to $\mathbb{Z}$ sending all generators to 1. A *cyclic n-fold cover* of $S^3 - K$ is the covering of $S^3 - K$ corresponding to the subgroup $\rho^{-1}(n\mathbb{Z})$ in $\pi_1(S^3 - K)$ and denoted $\widetilde{X}^n$. If $n = 0$, it is called an *infinite cyclic cover* of the knot complement $S^3 - K$ and denoted $\widetilde{X}^\infty$. The group of covering transformations is generated by an homeomorphism denoted T_n or T_∞.

We denote an open tubular neighborhood of K in S^3 by $N(K)$. We can assume that the closure $\overline{N(K)}$ of $N(K)$ in S^3 is a submanifold homeomorphic to the solid torus $D^2 \times S^1$. The complement $S^3 - N(K)$ is a strong deformation retract of $S^3 - K$ and a compact 3-manifold with boundary. The boundary of $S^3 - N(K)$, $\partial(S^3 - N(K))$, is a torus $S^1 \times S^1$ and it has a *preferred peripheral structure* that is defined as a choice of a pair of oriented simple closed curves called *meridian* and *longitude*. A *meridian* is a simple closed curve on $\partial(S^3 - N(K))$ which is oriented by the right-hand rule with respect to K, bounds a 2-disk in $\overline{N(K)}$, and carries a non zero element in $H_1(\partial(S^3 - N(K)))$. A *longitude* is a simple closed curve on $\partial(S^3 - N(K))$ that is isotopic to K in $\overline{N(K)}$ with orientation induced by K and null homologous in $S^3 - N(K)$.

1.3. Branched cyclic covers. For any positive integer n, the n-fold cyclic cover $\bar{X}^n$ of $S^3 - N(K)$ is defined similarly to the case $\widetilde{X}^n$ and it is a compact 3-manifold with boundary and a strong deformation retract of $\widetilde{X}^n$. The boundary of $\bar{X}^n$ is homeomorphic to a torus $S^1 \times S^1$ and has a preferred peripheral structure whose meridian and longitude are lifts of those in the base space $S^3 - N(K)$. An *n-fold cyclic cover of S^3 branched over K* is defined to be a 3-manifold M^n that is a union of a closed solid torus $D^2 \times S^1$ and $\bar{X}^n$ whose boundaries are glued together in such a way that the meridian and longitude of $\partial(D^2 \times S^1)$ are glued to the meridian and longitude of $\partial\bar{X}^n$, respectively, in an oriented manner. We can view $\widetilde{X}^n$ as a submanifold of M^n and its complement $M^n - \widetilde{X}^n$ is the branch set that is the lift of K. The n-fold cyclic cover $\widetilde{X}^n$ is the associated cover to M^n. We can also construct cyclic covers geometrically using a Seifert surface. For more details see Rolfsen [**Rol**].

1.4. Linking number. Let J and K be two disjoint oriented knots in S^3. The *linking number* of J and K, $lk(J, K)$, is defined as follows: Let $[J]$ denote the homology class in $H_1(S^3 - K)$ carried by J. Since $H_1(S^3 - K) \cong \mathbb{Z}$, we may choose a generator γ of this group carried by a meridian in a closed tubular neighborhood $\overline{N(K)}$

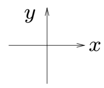
of K which is oriented by the right-hand rule. Write $[J] = n\gamma$. Define $lk(J, K) = n$. Note that the linking number can be defined even if J is a homology class in $S^3 - K$.

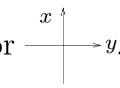
1.5. Seifert surface, form, and matrix. A *Seifert surface* of a knot K is an oriented surface F embedded in S^3 whose boundary is the knot K. The existence of a Seifert surface is well known. Since a Seifert surface has only one boundary component, the first homology of F is a free abelian group with $2g$ generators, where g is the genus of the surface F.

Let $i_+ : H_1(F) \to H_1(S^3 - F)$ denote the map which pushes a class off in the positive normal direction to the surface F by the right-hand rule and let i_- denote the map given by pushing off the other way. The *Seifert form* θ of a Seifert surface F is defined as follows: Let x and y be homology classes in $H_1(F)$. Then i_+y is a homology class in $H_1(S^3 - F)$. Choose a standard basis $\{a_1, \ldots, a_{2g}\}$ for a $\mathbb{Z}$-module $H_1(F)$ as shown in [**Rol**, p. 209]. We abuse the notation a_j to denote a simple closed curve carrying the class a_j in $H_1(F)$. The class x can be uniquely expressed as a linear combination of a's, $x = \sum_{j=1}^{2g} n_j a_j$. By inclusion $S^3 - F \to S^3 - a_j$, we consider i_+y as a class in $H_1(S^3 - a_j)$ and may define $lk(a_j, i_+y)$. Define $\theta(x, y) = \sum_{j=1}^{2g} n_j lk(a_j, i_+y)$. The form θ clearly depends on the choice of F.

For a basis $\{b_1, \ldots, b_{2g}\}$ for $H_1(F)$, the *associated Seifert matrix*, $B = (b_{i,j})$, is defined to be the $2g$ by $2g$ integral matrix with entries $b_{i,j} = \theta(b_i, b_j)$. Note that any two Seifert matrices A and B associated to a Seifert form are *congruent* to each other, that is, $A = PBP^t$ for an invertible integral matrix P. The *transpose* of a Seifert form θ is a quadratic form θ^t on $H_1(F)$ defined by $\theta^t(x, y) = \theta(y, x)$.

1.6. Intersection form. There is another bilinear form definable on the homology of a surface. If x and y are in $H_1(F)$, we may choose representative 1-cycles which intersect transversely. With an appropriate weighting of the intersections by

± 1, according to orientation conventions, their sum is the *intersection number*, denoted by $\langle x, y\rangle$. For example, one convention would be to view an intersection on the positive side of a surface and count $+1$ for 

and count -1 for 

. Note that this orientation convention is the opposite of the one given in Rolfsen [**Rol**, p. 202]. It is a well defined anti-symmetric bilinear form, called the *intersection pairing* or *intersection form*. By a Poincare duality, given a choice of basis, any matrix for the intersection form has determinant ± 1.

1.7. Algebraic Seifert form. The Seifert form and intersection form of a Seifert surface is related by a formula $\langle x, y\rangle = (\theta^t - \theta)(x, y)$. Hence any Seifert matrix A for a knot K in S^3 satisfies $\det(A^t - A) = \pm 1$. In general, an *algebraic* Seifert matrix is an integral matrix A satisfying $\det(A^t - A) = \pm 1$. An *algebraic* Seifert form is an integral valued bilinear form θ on finitely generated free $\mathbb{Z}$-module H such that a matrix for θ is a Seifert matrix, in other word, $\theta^t - \theta$ is *unimodular*, *i.e.*, the associated map $H \to \operatorname{Hom}(H, \mathbb{Z})$, defined by $x \mapsto (\theta^t - \theta)(x, \ )$, is an isomorphism. Observe that the rank of H must be even since the intersection form defined on H is an anti-symmetric unimodular form.

1.8. Homology of the infinite cyclic cover. Consider a knot K in S^3. Its complement $X = S^3 - K$ has an infinite cyclic covering space $\widetilde{X}^\infty$. Let Λ denote $\mathbb{Z}[t^{\pm 1}]$, the ring of (finite) Laurent polynomials with integer coefficients. We will describe the Λ-module structure of $H_1(\widetilde{X}^\infty)$. Choose a generator $T_\infty \colon \widetilde{X}^\infty \to \widetilde{X}^\infty$ of the group of covering translations. Define the product of an element $p(t)$ of Λ with an element α of $H_1(\widetilde{X}^\infty)$ by the formula $p(t)\alpha = p(T_\infty)\alpha$, where T_∞ is reused to denote the automorphism induced by T_∞ on the homology $H_1(\widetilde{X}^\infty)$.

Let F be a Seifert surface of K. The preimage of $S^3 - F$ in the infinite cyclic cover $\widetilde{X}^\infty$ has an infinite set of components, $\{X_i\}_{i\in\mathbb{Z}}$. A Mayer-Vietoris argument shows that $H_1(\widetilde{X}^\infty, \mathbb{Z})$ is generated by the homology groups of the components X_i. Furthermore, as a Λ-module, $H_1(\widetilde{X}^\infty, \mathbb{Z})$ is generated by $H_1(X_0, \mathbb{Z})$. The Mayer-Vietoris argument shows that with respect to the appropriate basis of $H_1(X_0, \mathbb{Z})$ the homology is presented by the matrix $A - tA^t$.

1.9. Alexander polynomial. The *Alexander polynomial* of K is defined to be $\Delta_K(t) = \det(A - tA^t)$, where A is a Seifert matrix. More generally, for any Seifert form θ, the Alexander polynomial is defined to be $\Delta_\theta(t) = \det(A - tA^t)$, where A is a matrix for θ. The Alexander polynomial is a knot invariant which is well-defined up to multiplication by units of Λ, *i.e.*, monomials $\pm t^n$, $n \in \mathbb{Z}$. One important property of $\Delta_K(t)$ is that multiplication by $\Delta_K(t)$ annihilates $H_1(\widetilde{X}^\infty, \mathbb{Z})$ which was first proved by Fox.

A polynomial f is said to be *reciprocal* if $f(t) = t^n f(t^{-1})$, where $n = \deg f$. The Alexander polynomial $\Delta_K(t)$ of a knot K is a reciprocal polynomial with $\Delta_K(1) = \pm 1$. The Alexander polynomial can be used to compute the order of the first homology of a q-fold cyclic cover of S^3 branched over a knot K, M^q. The following theorem was partially proved by Fox and a complete proof can be found in [**Web**].

THEOREM 2.1. $|H_1(M^q, \mathbb{Z})| = \prod_{i=1,\ldots,q-1} \Delta_K(\zeta_q^i)$, *where* ζ_q *denotes the* q*-th primitive root of unity and the equality holds if the right hand side is non zero.*

2. The knot concordance group

In this section we summarize definitions and results about the knot concordance group and the algebraic concordance group.

2.1. Definitions. We say a knot (S, K) is *slice* if it bounds a proper pair (B, D), where B is homeomorphic to the 4-ball B^4 and D is homeomorphic to the 2-disk B^2. The *mirror image* of a knot (S, K), denoted $-(S, K)$, is the image of (S, K) by an orientation reversing homeomorphism of S with the image of K reversed orientation. Knots (S_1, K_1) and (S_2, K_2) are called *concordant* (also known as *knot cobordant*) if $(S_1, K_1)\# -(S_2, K_2)$ is slice, where the connected sum is as usual for oriented pairs. The set of concordance classes of knots forms an abelian group under connected sum, denoted $\mathcal{C}$. The zero element is the class of the unknot and the inverse of a class of a knot K is the class of its mirror image.

Alternatively, knots (S_1, K_1) and (S_2, K_2) are called concordant if there is a pair $(S^3 \times [0,1], C)$, where C is homeomorphic to $S^1 \times [0,1]$ and $\partial(S^3 \times [0,1], C)$ is homeomorphic to the disjoint union $(S_1, K_1) \coprod -(S_2, K_2)$.

Working in the smooth category there is a similarly defined concordance group, $\mathcal{C}_s$. Since smooth submanifolds are locally flat, there is a homomorphism $\mathcal{C}_s \to \mathcal{C}$ that is surjective. It is not an isomorphism. It is known via gauge theory and Freedman's work that many topologically slice knots are not smoothly slice. For example, any knot with Alexander polynomial 1 is topologically slice [**FQ**], but the untwisted double of the right-handed trefoil (defined in Section 5) is not smoothly slice [**CoG**].

2.2. Algebraic concordance. An (algebraic) Seifert form θ defined on a $\mathbb{Z}$-module H is called *metabolic* or *null-concordant* if there is a $\mathbb{Z}$-submodule Z of H such that $2\,\mathrm{rank}(Z) = H$ and θ vanishes on Z, *i.e.*, $\theta(x, y) = 0$ for all $x, y \in Z$. Such a submodule Z is called a *metabolizer* of θ on H. Observe from the following lemma that such a submodule Z can be taken to be a direct summand of H. Throughout the rest of this chapter a metabolizer will be a direct summand unless stated otherwise.

LEMMA 2.2. *Let H be a finitely generated free module over a PID and let Z be a submodule of H. Then the submodule $Z' = \{v \in H \mid nv \in Z$ for a nonzero integer $n\}$ is a direct summand of H.*

PROOF. There is a short exact sequence $0 \to Z' \to H \to H/Z' \to 0$. From the definition of Z' we see that H/Z' is torsion free and hence free since the base ring is a PID. The exact sequence splits. Thus Z' is a direct summand of H. □

Observe also that if Z is a metabolizer of a Seifert from θ, Z is also a metabolizer of the associated intersection pairing $\theta^t - \theta$.

If θ_i is a Seifert form on H_i for each $i = 1, 2$, we denote by $\theta_1 \oplus \theta_2$ the Seifert form on $H_1 \oplus H_2$ given by $(\theta_1 \oplus \theta_2)(x \oplus u, y \oplus v) = \theta_1(x, y) + \theta_2(u, v)$, where $x, y \in H_1$ and $u, v \in H_2$. It is easy to see that $\theta_1 \oplus \theta_2$ is a Seifert form, *i.e.*, its matrix A satisfies $\det(A^t - A) = \pm 1$. We say that θ_1 and θ_2 are *concordant* if the Seifert form $\theta_1 \oplus -\theta_2$ on $H_1 \oplus H_2$ is metabolic.

Concordance of Seifert forms is an equivalence relation. Reflexivity and symmetry are both trivial. Transitivity immediately follows from the following cancellation lemma. It is due to Levine [**Lev2**] and the proof we give below is Kervaire's [**Ker**].

LEMMA 2.3. *If θ and η are Seifert forms and both $\theta \oplus \eta$ and η are null-concordant, then so is θ.*

PROOF. Let θ and η be defined on G and H, respectively. By assumption, there is a metabolizer Z of $\theta \oplus \eta$ in $G \oplus H$. Also, there is a metabolizer H_0 of η in H. Let $Z_0 = Z \cap (G \oplus H_0)$ and let G_0 be the image of Z_0 by the projection from $G \oplus H$ onto G. If $x, y \in G_0$, then there are $u, v \in H_0$ such that $x \oplus u$ and $y \oplus v$ are in Z. Then $\theta(x, y) = \theta(x, y) + \eta(u, v) = (\theta \oplus \eta)(x \oplus u, y \oplus v) = 0$. Hence θ vanishes on G_0.

It remains to prove that G_0 has the correct rank. We may assume H_0 is a direct summand of H. Then there is H_1 such that $H_0 \oplus H_1 = H$. Let Z_1 be the image of Z by the projection from $G \oplus H_0 \oplus H_1$ onto H_1. We have an exact sequence $0 \to Z_0 \to Z \to Z_1 \to 0$. Similarly there is an exact sequence $0 \to Z_0 \cap H \to Z_0 \to G_0 \to 0$.

These two exact sequences give:

$$\begin{aligned}\operatorname{rank} G_0 &= \operatorname{rank} Z_0 - \operatorname{rank}(Z_0 \cap H) \\ &= \operatorname{rank} Z - \operatorname{rank} Z_1 - \operatorname{rank}(Z_0 \cap H) \\ &= \frac{1}{2}\operatorname{rank} G + \frac{1}{2}\operatorname{rank} H - (\operatorname{rank} Z_1 + \operatorname{rank}(Z_0 \cap H)).\end{aligned}$$

We will prove that $\operatorname{rank} Z_1 + \operatorname{rank}(Z_0 \cap H) \leq \frac{1}{2}\operatorname{rank} H$. Observe that $Z_0 \cap H = Z \cap H_0$. First, we will show that $H_0 \oplus Z_1$ and $Z \cap H_0$ are orthogonal under $\eta - \eta^t$. Since H_0 is a metabolizer of η, H_0 and $Z \cap H_0$ are orthogonal under $\eta - \eta^t$. Let $x \in Z_1$ and $y \in Z \cap H_0$. There are $u \in G$ and $z \in H_0$ such that $u \oplus (x + z) \in Z$. Observe that $(\eta - \eta^t)(z, y) = 0$ since $z, y \in H_0$. We have $(\eta - \eta^t)(x, y) = (\eta - \eta^t)(x + z, y) = (\theta - \theta^t)(u, 0) + (\eta - \eta^t)(x + z, y) = (\theta \oplus \eta)(u \oplus (x + z), y) - (\theta^t \oplus \eta^t)(u \oplus (x + z), y) = 0$ since $u \oplus (x + z)$ and y are in Z. Hence, $H_0 \oplus Z_1$ and $Z \cap H_0$ are orthogonal under $\eta - \eta^t$. Since $\eta - \eta^t$ is unimodular, $\operatorname{rank}(H_0 \oplus Z_1) + \operatorname{rank}(Z \cap H_0) \leq \operatorname{rank} H$. Since $\operatorname{rank}(H_0 \oplus Z_1) = \operatorname{rank} H_1 + \operatorname{rank} Z_1$ and $\operatorname{rank} H_0 = \frac{1}{2}\operatorname{rank} H$,

$$\operatorname{rank} Z_1 + \operatorname{rank}(Z \cap H_0) \leq \frac{1}{2}\operatorname{rank} H.$$

Combining the above equality and inequality we have an inequality

$$\begin{aligned}\operatorname{rank} G_0 &= \frac{1}{2}\operatorname{rank} G + \frac{1}{2}\operatorname{rank} H - (\operatorname{rank} Z_1 + \operatorname{rank}(Z_0 \cap H)) \\ &\geq \frac{1}{2}\operatorname{rank} G.\end{aligned}$$

The equality must hold since $\theta - \theta^t$ is unimodular and vanishes on G_0. This completes the proof. □

2.3. Algebraic concordance group. Let $\mathcal{G}$ be the set of concordance classes of algebraic Seifert forms. The direct sum of Seifert forms induces an addition of the concordance classes and turns the set $\mathcal{G}$ into an abelian group.

A matrix A for a metabolic Seifert form θ is congruent to a matrix of the form $\begin{pmatrix} 0 & A_1 \\ A_2 & A_3 \end{pmatrix}$ where A_i are square matrices of the same size. Such a matrix A is called *null-concordant*. We can define concordance of Seifert matrices along the lines of concordance of Seifert forms. The set of concordance classes of Seifert matrices forms an abelian group and it is isomorphic to $\mathcal{G}$. This group is classified by Levine [**Lev1**]. We denote by $\mathbb{Z}/n$ a cyclic group of order n.

Theorem 2.4 (Levine). $\mathcal{G} = \left(\overset{\infty}{\oplus}\mathbb{Z}/2\right) \oplus \left(\overset{\infty}{\oplus}\mathbb{Z}/4\right) \oplus \left(\overset{\infty}{\oplus}\mathbb{Z}\right)$, *that is, the group* $\mathcal{G}$ *is isomorphic to a direct sum of cyclic groups of orders 2, 4 and* ∞, *and there are an infinite number of summands of each of these orders.*

If a knot K is slice then for any Seifert surface F its Seifert form is metabolic. This defines a well-defined map from $\mathcal{C}$ to $\mathcal{G}$. Levine [**Lev2**] proved that this map is surjective.

Theorem 2.5 (Levine). *The map* $K \mapsto \theta$ *induces a well defined surjective homomorphism from* $\mathcal{C}$ *to* $\mathcal{G}$.

A knot is called *algebraically slice* if its Seifert form is null-concordant. The group $\mathcal{G}$ is called Levine's *algebraic concordance group.*

We can also define an algebraic concordance group over a field F. Consider a F-valued bilinear form θ on a finitely generated vector space V over the field F. We say the form θ is *admissible* if $\theta - \theta^t$ and $\theta + \theta^t$ are non-singular, *i.e.*, their matrices for a choice of a basis of V have non zero determinants. Defining concordance as for Seifert forms, it follows similarly that concordance is an equivalence relation among admissible matrices and the set of concordance classes, $\mathcal{G}_F$, becomes an abelian group under direct sum. We will only consider the case $F = \mathbb{Q}$.

There is an obvious homomorphism $\mathcal{G} \to \mathcal{G}_{\mathbb{Q}}$, since, for an integral Seifert form θ, $\theta - \theta^t$ unimodular implies a matrix for $\theta + \theta^t$ has odd determinant. This is, in fact, an injective homomorphism, since, by the argument of Levine [**Lev2**, Lemma 8], an integral matrix is null-concordant over the integers if and only if it is null-concordant over the rationals.

2.4. Non-singular Seifert form. Let θ denote the Seifert form. Let A denote the Seifert matrix for θ with respect to some basis $\{a_1, \ldots, a_{2g}\}$ for $H_1(F)$. With respect to this basis the intersection form is given by the matrix $A^t - A$. Let $\{\alpha_1, \ldots, \alpha_{2g}\}$ denote the basis for $H_1(S^3 - F)$ such that $lk(a_i, \alpha_j) = \delta^i_j$ as in Rolfsen [**Rol**, p. 209]. Then i_+ with respect to these bases is given by A. Define $j : H_1(S^3 - F) \to H_1(F)$ by $\langle jx, y \rangle = lk(x, y)$. The homomorphism j is given by the matrix $(A - A^t)^{-1}$ with respect to the above basis. An integral form θ on H is said to be *non-singular* if its associated homomorphism $H \to \mathrm{Hom}(H, \mathbb{Z})$, defined by $x \mapsto \theta(x, \ )$, is injective, in other word, a matrix for θ has nonzero determinant. Levine [**Lev1**, Lemma 8] proved that every algebraic concordance class contains a non-singular representative.

PROPOSITION 2.6. *Every Seifert form is concordant to a non-singular one.*

PROOF. Let θ be a Seifert form on H. Suppose θ is singular, *i.e.*, there is $a_1 \in H$, $a_1 \neq 0$, such that $\theta(a_1, x) = 0$ for all $x \in H$. We may assume that a_1 is not divisible in H, namely, there is no nonzero element $v \in H$ such that $a_1 = mv$ for an integer m with $|m| \geq 2$. Then by Lemma 2.2 the submodule generated by a_1, $\mathbb{Z}a_1$, is a direct summand in H. Since $\langle\ ,\ \rangle = \theta^t - \theta$ is unimodular, there is $a_2 \in H$ such that $\langle a_1, a_2 \rangle = 1$. Observe that $1 = \langle a_1, a_2 \rangle = \theta(a_2, a_1) - \theta(a_1, a_2) = \theta(a_2, a_1)$. Let H_0 be the submodule of H generated by a_1 and a_2. We will show that $H_0 = \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$. Suppose $x \in \mathbb{Z}a_1 \cap \mathbb{Z}a_2$. Then there are integers m_i, $i = 1, 2$, such that

$x = m_1 a_1 = m_2 a_2$. We have that $0 = \langle m_1 a_1, a_1 \rangle = \langle m_2 a_2, a_1 \rangle = -m_2$. Thus, $x = 0$ and $H_0 = \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$.

Let G be the orthogonal complement of a_1 and a_2 under $\langle\ ,\ \rangle$, namely, $G = \{x \in H \mid \langle a_1, x\rangle = \langle a_2, x\rangle = 0\}$. We will show that $H = H_0 \oplus G$. Suppose that $x \in H_0 \cap G$. Then $x = m_1 a_1 + m_2 a_2$ for some integers m_1 and m_2. Observe that $0 = \langle a_1, x\rangle = \langle a_1, m_2 a_2\rangle = m_2$ and $0 = \langle a_2, x\rangle = \langle a_2, m_1 a_1\rangle = -m_1$. This implies that $x = 0$ and $H_0 \cap G = \{0\}$. To show $H_0 + G = H$, let y be any element in H and consider $x = y - \langle y, a_2\rangle a_1 + \langle y, a_1\rangle a_2$. Note that $\langle a_1, x\rangle = \langle a_1, y\rangle + \langle y, a_1\rangle\langle a_1, a_2\rangle = 0$ and $\langle a_2, x\rangle = \langle a_2, y\rangle - \langle y, a_2\rangle\langle a_2, a_1\rangle = 0$. Hence $x \in G$ and $y \in H_0 + G$. This shows that $H = H_0 + G$. It follows that $H = H_0 \oplus G$.

Let η be the restriction of θ on G. We will see that η is a Seifert form on G which is concordant to θ. The form η is a Seifert form on G since $\eta^t - \eta$ is the restriction of $\langle\ ,\ \rangle = \theta^t - \theta$ on G. To show η is concordant to θ, we let $D = \{(x, x) \in G \oplus G \mid x \in G\}$ and consider $\mathbb{Z}a_1 \oplus D$ in $H \oplus G$. It is easy to see that $2\,\mathrm{rank}(\mathbb{Z}a_1 \oplus D) = \mathrm{rank}(H \oplus G)$. Next, we will show that the form $\theta \oplus -\eta$ on $H \oplus G$ vanishes on $\mathbb{Z}a_1 \oplus D$. For $x, y \in G$ and $m, n \in \mathbb{Z}$,

$$\begin{aligned}(\theta \oplus -\eta)((ma_1 + x) \oplus x, (na_1 + y) \oplus y) &= \theta(ma_1 + x, na_1 + y) - \eta(x, y)\\ &= n\theta(x, a_1) + \theta(x, y) - \theta(x, y)\\ &= n\theta(x, a_1) - n\theta(a_1, x)\\ &= n\langle a_1, x\rangle\\ &= 0.\end{aligned}$$

Thus $\mathbb{Z}a_1 \oplus D$ is a metabolizer for $\theta \oplus -\eta$ and hence θ is concordant to η. The lemma follows by the induction on the rank of H. □

2.5. S-equivalence. From the observation given in the proof of Proposition 2.6, Levine [**Lev1**] and Trotter [**Tro**] introduced S-equivalence for Seifert matrices. The relation of *S-equivalence* is defined as follows: Let A and B be two Seifert matrices. A is (integrally) *congruent* to B if $A = PBP^t$ for some integral matrix P with $\det(P) = \pm 1$. A is a *row enlargement* of B, and B is a *row reduction* of A, if A is obtained by bordering B with two additional rows and columns in such a way that

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & u \\ 0 & v & B \end{pmatrix},$$

where x is a number, v is a column vector, and u is a row vector. A is a *column enlargement* of B, and B is a *column reduction* of A if A^t is a row enlargement of B^t. *S-equivalence* is defined as the equivalence relation generated by congruence, enlargement, and reduction. It is easy to check that the class of Seifert matrices is closed under S-equivalence. (Note that x and u in the above matrix can be annihilated by a congruence transformation.) A basic result concerning S-equivalence states that two Seifert matrices for a given knot are S-equivalent. This was proved by Murasugi [**Mu**].

2.6. Isometric structure. Following Kervaire [**Ker**] and Stoltzfus [**Sto1**], we define *the associated isometric structure* $s\colon H_1(F) \to H_1(F)$ by the equation $\theta(y,x) = \langle sx, y\rangle$. Observe that $\langle x,y\rangle = \theta(y,x) - \theta(x,y) = \langle sx,y\rangle - \langle sy,x\rangle = \langle sx,y\rangle + \langle x,sy\rangle$. We see that s is $j \circ i_+$ and is given by the matrix $G = (A - A^t)^{-1}A$. Note $s-1$ is given by $G - I = (A - A^t)^{-1}A - I = (A-A^t)^{-1}A^t$ and $s-1$ is actually $j \circ i_-$. Observe that the characteristic polynomial of s is the polynomial $x^{2g}\Delta_\theta(1-x^{-1})$, where $\Delta_\theta(t)$ is the Alexander polynomial of the Seifert form θ with degree $2g$.

The map s is called an isometric structure in the following sense: Suppose that s is non-singular. Then the map $t = 1 - s^{-1}\colon H \otimes \mathbb{Q} \to H \otimes \mathbb{Q}$ is an automorphism. (It is easily verified that t has a matrix $A^{-1}A^t$, where A is a matrix for the Seifert form θ.) Moreover, t is an isometry for θ and $\langle\ ,\ \rangle$ extended to $H \otimes \mathbb{Q}$. The Alexander polynomial $\Delta_\theta(t) = \det(A - tA^t)$ is the characteristic polynomial of t.

We can extend the notion of isometric structure to an algebraic object. An (algebraic) *isometric structure* is a pair $(\langle\ ,\ \rangle, s)$, where $\langle\ ,\ \rangle$ is an anti-symmetric unimodular bilinear form on a finitely generated free $\mathbb{Z}$-module and s is a linear endomorphism satisfying $\langle x, y\rangle = \langle sx, y\rangle + \langle x, sy\rangle$. A *metabolizer* of an isometric structure $(\langle\ ,\ \rangle, s)$ is an s-invariant metabolizer of $\langle\ ,\ \rangle$. The concordance of isometric structures can be defined as in algebraic concordance. It was proved by Levine [**Lev1**] and Kervaire [**Ker**] that the set of concordance classes of isometric structures is an abelian group isomorphic to the algebraic concordance group $\mathcal{G}$. In fact, the metabolizer of a Seifert form is a metabolizer of its associated isometric structure. For simplicity, we will denote an isometric structure only by s.

2.7. Splitting of metabolizer. The following lemma is a refinement of Proposition 3 of Kervaire [**Ker**, p. 94]. Its geometrical origins are in the work of Levine [**Lev1**] on the concordance group of knots. Here the splitting of a metabolizer is emphasized. They did not need the splitting of a metabolizer, while we will need the splitting to prove Theorem 1.1. We say two elements f and g in a PID R are *relatively prime* or *coprime* if there are u and v in R such that $uf + vg = 1$.

LEMMA 2.7. *Let θ_1 and θ_2 be Seifert forms on H_1 and H_2. Suppose that their Alexander polynomials are relatively prime in $\mathbb{Q}[t^{\pm 1}]$ and that either θ_1 or θ_2 is non-singular. If $\theta_1 \oplus \theta_2$ is null-concordant with a metabolizer Z, then θ_1 and θ_2 are null-concordant with metabolizers Z_1 and Z_2, respectively, such that $Z_i = Z \cap H_i$, $i = 1, 2$, and $Z = Z_1 \oplus Z_2$.*

PROOF. As stated earlier, we automatically assume that Z is a direct summand of $H = H_1 \oplus H_2$. Consider the associated isometric structures s_i to θ_i, $i = 1, 2$. Then $s_1 \oplus s_2$ on H is the associated isometric structure s to $\theta_1 \oplus \theta_2$. Let $Z_i = Z \cap H_i$,

$i = 1, 2$, and let $\varphi_i(x) = x^{2g_i}\Delta_i(1 - x^{-1})$, where $2g_i$ is the rank of H_i and Δ_i is the Alexander polynomial of θ_i. Note that $\varphi_i(x)$ lies in $\mathbb{Z}[x]$.

We will show that there are u_1 and u_2 in $\mathbb{Z}[x]$ and a non zero integer c such that $u_1\varphi_1 + u_2\varphi_2 = c$. Since it is assumed that Δ_1 and Δ_2 are relatively prime in $\mathbb{Q}[t^{\pm 1}]$ there are v_1' and v_2' in $\mathbb{Q}[t^{\pm 1}]$ such that $v_1'\Delta_1 + v_2'\Delta_2 = 1$. Replacing t with $1 - x^{-1}$ we see that there are v_1 and v_2 in $\mathbb{Q}[x^{\pm 1}]$ such that $v_1\varphi_1 + v_2\varphi_2 = 1$. There is a nonnegative integer n for which $x^n v_i \in \mathbb{Q}[x]$ for $i = 1, 2$. Multiplication by x^n gives the equality

$$x^n v_1\varphi_1 + x^n v_2\varphi_2 = x^n$$

in $\mathbb{Q}[x]$. Without loss of generality assume that θ_1 is non-singular. Then Δ_1 has a nonzero constant coefficient and so does φ_1. This implies that φ_1 and x^n are relatively prime in $\mathbb{Q}[x]$, that is, there are w_1 and w_2 in $\mathbb{Q}[x]$ such that

$$w_1\varphi_1 + w_2 x^n = 1.$$

Combining the above two equalities we have $w_2x^nv_1\varphi_1 + w_2x^nv_2\varphi_2 = w_2x^n = 1 - w_1\varphi_1$ or

$$(w_1 + w_2x^nv_1)\varphi_1 + w_2x^nv_2\varphi_2 = 1.$$

Since $w_1 + w_2x^nv_1$ and $w_2x^nv_2$ are polynomials in $\mathbb{Q}[x]$ there is a non zero integer c such that $c(w_1 + w_2x^nv_1)$ and $cw_2x^nv_2$ lie in $\mathbb{Z}[x]$. This shows that $u_1\varphi_1 + u_2\varphi_2 = c$ for some polynomials u_1 and u_2 in $\mathbb{Z}[x]$ and a non zero integer c. We will no longer assume that θ_1 is non-singular. In the rest of the proof either θ_1 or θ_2 is non-singular.

Let z be an element in Z. There are $z_1 \in H_1$ and $z_2 \in H_2$ with $z = z_1 + z_2$. We will show that $z_i \in Z_i$ for $i = 1, 2$. To do this it suffices to prove that z_i lies in Z. Each φ_i is the characteristic polynomial for s_i, and so $\varphi_i(s_i) = 0$. Using this and

$s(z_i) = s_i(z_i)$, we have

$$\begin{aligned} cz_1 &= u_1(s)\varphi_1(s)z_1 + u_2(s)\varphi_2(s)z_1 \\ &= u_1(s_1)\varphi_1(s_1)z_1 + u_2(s_1)\varphi_2(s_1)z_1 \\ &= u_2(s_1)\varphi_2(s_1)z_1 \\ &= u_2(s_1)\varphi_2(s_1)z_1 + u_2(s_2)\varphi_2(s_2)z_2 \\ &= u_2(s)\varphi_2(s)z. \end{aligned}$$

Since Z is s-invariant, $cz_1 = u_2(s)\varphi_2(s)z \in Z$, and since Z is a direct summand, this implies $z_1 \in Z$, and hence $z_1 \in Z_1$. Similarly, $z_2 \in Z_2$. So $Z = Z_1 + Z_2$. This shows that $Z = Z_1 \oplus Z_2$ since $H = H_1 \oplus H_2$ and $Z_i = Z \cap H_i$.

Since Z is s-invariant, each Z_i is s_i-invariant. Since $H/Z = H_1/Z_1 \oplus H_2/Z_2$ is torsion free, each Z_i is a direct summand of H_i. Since the intersection pairing $\langle\ ,\ \rangle$ on H splits, we see that $2 \operatorname{rank} Z_i = \operatorname{rank} H_i$. Thus each Z_i is a metabolizer for s_i and hence θ_i. □

Combining Proposition 2.6 and Lemma 2.7, we have a polynomial splitting of Seifert forms.

THEOREM 2.8 (Levine [**Lev1**]). *If K_1 and K_2 are algebraically slice with relatively prime Alexander polynomials, then K_1 and K_2 are both algebraically slice.*

2.8. Splitting of algebraic concordance group. Let $\lambda \in \mathbb{Z}[x]$ be a reciprocal polynomial of even degree with $\lambda(1) = 1$. Define an *isometric structure* over $\mathbb{Z}$ or $\mathbb{Q}$ *associated with* λ to be an anti-symmetric unimodular bilinear form $\langle\ ,\ \rangle$ on a $\mathbb{Z}$-space (resp. $\mathbb{Q}$-space) H together with a homomorphism $s\colon H \to H$ satisfying

(1) $\langle sx, y\rangle + \langle x, sy\rangle = \langle x, y\rangle$,
(2) $\varphi(x) = x^{2g}\lambda(1 - x^{-1})$ is the minimal polynomial of s.

We can also define concordance of isometric structures associated with λ as before. The set of equivalence classes is denoted by $\mathcal{G}^\lambda$, respectively $\mathcal{G}^\lambda_{\mathbb{Q}}$, depending on the underlying ring. They are abelian groups under direct sum.

Levine's polynomial splitting of algebraic concordance can be restated in the following proposition. It was proved by Levine and Kervaire. For details, see Kervaire [**Ker**] and Stoltzfus [**Sto2**].

Proposition 2.9. *There are injections*

$$\sum_\lambda \mathcal{G}^\lambda \longrightarrow \mathcal{G} \longrightarrow \sum_\lambda \mathcal{G}^\lambda_{\mathbb{Q}},$$

where direct sums run over all irreducible, reciprocal $\lambda \in \mathbb{Z}[x]$ of even degree satisfying $\lambda(1) = 1$.

It is known that both injections in the previous proposition are not surjective. In contrary, if the underlying ring is rational, the rational algebraic concordance group, $\mathcal{G}_{\mathbb{Q}}$, splits completely. It was proved by Levine [**Lev1**].

Theorem 2.10 (Levine).

$$\mathcal{G}_{\mathbb{Q}} = \sum_\lambda \mathcal{G}^\lambda_{\mathbb{Q}}.$$

2.9. Witt groups. A Witt group is a kind of generalization of the algebraic concordance group, but defined on symmetric forms. Let R be a commutative ring with involution $\mathcal{J}$. To simplify algebra, we will assume that R is a PID. The main examples will be the integers, $\mathbb{Z}$, the rational field, $\mathbb{Q}$, and the field of rational functions on the complex numbers $\mathbb{C}$, $\mathbb{C}(t)$, with involution induced by complex conjugation and the map that sends t to t^{-1}.

We consider unimodular symmetric hermitian forms, θ, on free finite rank R-modules, H. Symmetry means that $\theta(x, y) = \mathcal{J}(\theta(y, x))$; unimodular means that the associated map $H \to \operatorname{Hom}(H, R)$ given by $x \mapsto \theta(x, \ )$ is a (skew linear) isomorphism;

hermitian means that θ is linear in the first variable and skew linear in the second variable, that is, $\theta(x, ay) = \mathfrak{J}(a)\theta(x, y)$. Such a form is called *metabolic* if there is a submodule Z of H, with $\text{rank}(H) = 2\,\text{rank}(Z)$, on which θ is trivial. Forms θ_1 and θ_2 are *concordant* if the direct sum $\theta_1 \oplus -\theta_2$ is metabolic. The set of equivalence classes is called the *Witt group* of R, denoted $W(R)$. Under the operation induced by direct sum, $W(R)$ forms an abelian group.

2.10. Tristram-Levine signature. Let $A(t)$ be a non-singular hermitian matrix with coefficients in $\mathbb{C}(t)$. There is a Witt group of such matrices, $W(\mathbb{C}(t))$. For a unit complex number ω one has the signature $\sigma(A(\omega))$. Though $A(t)$ is assumed non-singular, $A(\omega)$ may be singular, and hence, even if $A(t)$ is metabolic, $\sigma(A(\omega))$ may be nonzero. For this reason, taking signatures at unit numbers ω is not a well defined function on the Witt group $W(\mathbb{C}(t))$. However, the limit $\lim_{s\to 0}(\sigma(A(\omega e^{si})) + \sigma(A(\omega e^{-si}))/2$ does yield a well defined *signature function*, σ_ω, on $W(\mathbb{C}(t))$.

The *Tristram-Levine signature function* of a knot K, $\sigma_\omega(K)$, is defined to be the signature function of the form $A_K(t) = (1-t)A_K + (1-t^{-1})A_K^t$, where A_K is a Seifert matrix for K. To simplify notation we have $\sigma_r(K) = \sigma_\omega(K)$, where r is a real number and $\omega = e^{2\pi i r}$. The following states some basic results of the signature function. A proof can be found in [**Ma**].

Proposition 2.11. *(a) The hermitian form $A_K(\omega)$ with ω a unit complex number is singular only if $\omega = 1$ or $\Delta_K(\omega) = 0$.*

(b) If the root of Δ_K is of multiplicity one the jump of the signature function is 0 *or* ± 2.

2.11. Satellite knot and Litherland result. Let K be a knot in S^3. By an *axis* for K of *winding* number w we mean an unknotted simple closed curve γ in

$S^3 - K$ having linking number w with K. Let V be a solid torus complementary to a tubular neighborhood of γ, with K contained in the interior of V. There is a preferred generator v for $H_1(V)$, specified by the condition $lk(v, \gamma) = +1$. For any knot C in S^3 there is an untwisted orientation-preserving embedding $h\colon V \to S^3$ taking V onto a tubular neighborhood of C such that C represents $h_*(v)$ in $H_1(hV)$. We say that the knot $h(K)$, denoted $C(K)$, is a *satellite* of C with *orbit* K, *axis* γ, and *winding number* w.

The following is Theorem 2 of Litherland [**Lit1**].

Theorem 2.12. [**Lit1**] *Let $C(K)$ be a satellite of C with orbit K and winding number w. Then*

$$\sigma_r(C(K)) = \sigma_{wr}(C) + \sigma_r(K).$$

2.12. Linking form. Let M be a rational homology sphere. There is a *linking form* $\ell : H_1(M;\mathbb{Z}) \times H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ defined as follows. For elements $x, y \in H_1(M;\mathbb{Z})$, pick disjoint representative cycles, which we will again denote x and y. Since y is finite order, we can find a 2-chain z with $\partial(z) = ry$ for some integer r. Define $\ell(x, y) = (x \cdot z)/r$, where $x \cdot z$ denotes the intersection number of chains. One can show that this is well-defined and symmetric.

There is a homological interpretation of the linking form. Via duality one has that $H_1(M;\mathbb{Z}) = H^2(M;\mathbb{Z})$. By the universal coefficient theorem, we have $H^2(M;\mathbb{Z}) = \mathrm{Ext}(H_1(M;\mathbb{Z}), \mathbb{Z})$. From the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ we have that $\mathrm{Ext}(H_1(M;\mathbb{Z}), \mathbb{Z}) = \mathrm{Hom}(H_1(M;\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. Putting this together we have that $H_1(M;\mathbb{Z}) = \mathrm{Hom}(H_1(M;\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. A careful examination of the sequence of isomorphisms constructed above shows that this isomorphism is given by the map $x \mapsto \ell(x, \cdot)$. This implies that the linking form is unimodular, meaning that the induced map $H_1(M;\mathbb{Z}) \to \mathrm{Hom}(H_1(M;\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ is an isomorphism. Using this we will identify a class of order q in $H_1(M;\mathbb{Z})$ with *characters* on $H_1(M;\mathbb{Z})$ taking values in $\mathbb{Z}/q \subset \mathbb{Q}/\mathbb{Z}$.

3. Casson-Gordon invariant

In this section we define Casson-Gordon invariants of a closed 3-manifold and of a knot.

3.1. Casson-Gordon invariants of a closed 3-manifold. Let $(M, \bar{\chi})$ be a closed 3-manifold with a homomorphism

$$\bar{\chi}\colon H_1(M) \to \mathbb{Z}/q \oplus \mathbb{Z}.$$

The bordism group $\Omega_3(\mathbb{Z}/q \oplus \mathbb{Z})$ is finite, so $r(M, \bar{\chi}) = \partial(W^4, \bar{\phi})$, where r is an integer, W^4 is a 4-manifold and $\bar{\phi}\colon H_1(W^4) \to \mathbb{Z}/q \oplus \mathbb{Z}$.

According to Wall [**Wal**], there is an intersection pairing $t(W^4, \bar{\phi})\colon H_2(W^4; \mathbb{C}(t)) \times H_2(W^4; \mathbb{C}(t)) \to \mathbb{C}(t)$, where the field coefficients are twisted by the $\mathbb{Z}$ action given by multiplication by t and the $\mathbb{Z}/q$ action given by multiplication by $e^{2\pi i/q}$. This is hermitian with respect to the involution sending $p(t)$ to $\bar{p}(t^{-1})$. Casson and Gordon showed that it is non-singular if q is a prime power p^n and M is a $\mathbb{Z}/p$-homology circle. The intersection pairing $t(W^4, \bar{\phi})$ is viewed as an element in the Witt group $W(\mathbb{C}(t))$.

Let $t_0(W^4)$ be the image of the intersection pairing on $H_2(W^4;\mathbb{Q})$ (with untwisted coefficients) in $W(\mathbb{C}(t))$. The invariant τ is defined by

$$\tau(M^3, \bar{\chi}) = \frac{1}{r}\left(t(W^4, \bar{\phi}) - t_0(W^4)\right) \in W(\mathbb{C}(t)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Casson and Gordon showed that the invariant is independent of r and W^4.

3.2. Casson-Gordon invariants of a knot. Let K be a knot in S^3, M_K^q its q-fold branched cover, and χ a character from $H_1(M_K^q)$ to $\mathbb{Z}/q$. Throughout this section, assume q is a power of prime.

Let $S_{K,0}$ be the closed 3-manifold obtained from S^3 by performing 0-surgery on K. We write $M_{K,0}$ for the q-fold cyclic covering of $S_{K,0}$. Then there is a canonically induced character $\bar{\chi}\colon H_1(M_{K,0}) \to \mathbb{Z}/q \oplus \mathbb{Z}$. This follows from the fact that $H_1(M_{K,0})$ canonically splits as $H_1(M_K^q) \oplus \mathbb{Z}$ with the generator of the $\mathbb{Z}$ factor given by the meridian of the lift of K. Hence $\bar{\chi}$ is defined by mapping the meridian to $(0,1) \in \mathbb{Z}/q \oplus \mathbb{Z}$. The *Casson-Gordon invariant* τ of a knot K in S^3 and character χ is defined by

$$\tau(K,\chi) = \tau(M_{K,0}, \bar{\chi}) \in W(\mathbb{C}(t)) \otimes \mathbb{Q}.$$

Recall that the generator of the group of covering transformations of M_K^q is denoted T_q; T_q is of order q. The linking form on $H_1(M_K^q;\mathbb{Z})$ is invariant under the group of covering transformations. The main result of [**CaG**] is the following.

THEOREM 2.13 (Casson and Gordon). *Let q be a prime power and p a prime. If K is slice there is a subgroup Z in $H_1(M_K^q;\mathbb{Z})$ such that $|Z|^2 = |H_1(M_K^q;\mathbb{Z})|$, the linking form on M_K^q vanishes on Z, and, for any character χ with values in $\mathbb{Z}/p^r$ that vanishes on Z, $\tau(K,\chi) = 0$. Furthermore, Z is invariant under the action of T_q.*

3.3. Casson-Gordon signature invariant. The associated signature invariant to a Casson-Gordon invariant is defined as follows. For a class in $W(\mathbb{C}(t))$ the signature is defined by evaluating a representative of the class at a unit complex number and taking the limit of the signature of the resulting complex valued form as the unit complex number approaches 1. This map induces a homomorphism $\sigma_1 : W(\mathbb{C}(t)) \otimes \mathbb{Q} \to \mathbb{Q}$. The *Casson-Gordon signature invariant* σ for a 3-manifold M is defined by $\sigma(M,\bar{\chi}) = \sigma_1(\tau(M,\bar{\chi}))$, and for a knot K, $\sigma(K,\chi) = \sigma_1(\tau(K,\chi))$. Note, in [**CaG**] this $\sigma(K,\chi)$ is denoted $\sigma_1\tau(K,\chi)$.

3.4. Properties of Casson-Gordon invariants. We summarize the properties of Casson-Gordon invariants we will need. A result concerning the Casson-Gordon invariant is that $\tau(K,\chi) = \tau(K,\chi \circ T_q)$ where T_q denotes a generator of the group of covering translations, *i.e.*, the group of covering translations preserves the Casson-Gordon invariants. A simple result concerning the signature invariant is that $\sigma(K,\chi) = \sigma(K,-\chi)$. This follows from the fact that signatures are unchanged under complex conjugation.

A much deeper result is the additivity result proved by Gilmer [**G1**] (case $q = 2$) and Litherland [**Lit2**] (case q a prime power), as we describe. If $K = K_1 \# K_2$, we have $M_K^q = M_{K_1}^q \# M_{K_2}^q$, $H_1(M_K^q;\mathbb{Z}) = H_1(M_{K_1}^q;\mathbb{Z}) \oplus H_1(M_{K_2}^q;\mathbb{Z})$ and any character χ on $H_1(M_K^q;\mathbb{Z})$ can be written as $\chi_1 \oplus \chi_2$ with χ_i a character on $H_1(M_{K_i}^q;\mathbb{Z})$. The Casson-Gordon invariant of K is the sum of those of K_1 and K_2.

Theorem 2.14. $\tau(K,\chi) = \tau(K_1,\chi_1) + \tau(K_2,\chi_2)$.

Another result concerning the Casson-Gordon invariant concerns its value at the trivial character, $\tau(K,0)$. The result was first proved by Litherland [**Lit2**, Corollary B2].

Theorem 2.15. $\tau(K,0) = 0$ *if* K *is algebraically slice.*

3.5. Casson-Gordon invariants of a satellite knot. Litherland [**Lit2**] described the relationship between the Casson-Gordon invariants of a satellite knot and those of its component parts. We will describe these results only in the special case that we will need.

Let $C(K)$ be a satellite of C with axis γ, orbit K, and winding number 0. (For definition of a satellite knot, see Subsection 2.11.) Construct the union $(S^3 - N(\gamma)) \cup (S^3 - N(C))$ where the union is formed via a homeomorphism that interchanges meridians and longitudes. The resulting manifold is easily seen to be homeomorphic to S^3; it is the union of a knot complement with a solid torus. A fairly simple geometric argument shows that the image of K in the union represents the satellite $C(K)$.

To understand the effect of this interpretation of a satellite on covers and the associated invariants of the knots, let M^q_K be the q-fold branched cover of K. The unknotted curve γ lifts to a set of distinct curves, $\{\tilde{\gamma}_0, T_q(\tilde{\gamma}_0), \ldots, T_q^{q-1}(\tilde{\gamma}_0)\} = \{\tilde{\gamma}_i\}_{i=0,\ldots,q-1}$, in M^q_K. Thus, $M_{C(K)}$ is constructed from M^q_K by removing neighborhoods of each $\tilde{\gamma}_i$ and replacing them with copies of the complement of C. Since we are simply removing a homology circle and replacing it with another homology circle via maps that are homologically identical, the construction leaves the homology of the space unchanged and there is a natural correspondence between the homology, and cohomology, groups of M^q_K and $M^q_{C(K)}$. In particular we can identify a character χ on $H_1(M^q_{C(K)};\mathbb{Z})$ with a character also denoted χ on $H_1(M^q_K;\mathbb{Z})$. In this situation the Casson-Gordon invariants of $C(K)$ are related to the Tristram-Levine signatures of C as described in [**Lit2**]. (In the following theorem the character χ is assumed to have images into $\mathbb{Q}/\mathbb{Z}$ by the homomorphism $\mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ sending $1 \mapsto \frac{1}{n}$.)

Theorem 2.16.

$$\sigma(C(K),\chi) = \sigma(K,\chi) + \sum_{i=0}^{q-1} \sigma_{\chi(\tilde{\gamma}_i)}(C).$$

3.6. Gilmer's enhancement of Casson-Gordon invariants. Gilmer [**G2**] combined the slicing obstructions of Levine with those of Casson and Gordon in a nontrivial way. Essentially he related the metabolizer for the Seifert form of a slice knot to the characters on the homology of the q-fold branched cover for q a prime power.

Let K be a knot in the 3-sphere S^3 with Seifert surface F with intersection pairing $\langle\ ,\ \rangle$ and isometric structure s. Let M^q denote the q-fold branched cyclic cover of S^3 along K. Then, by Seifert, $G^q - (G - I)^q$ is a presentation matrix for $H_1(M^q)$. Compare Lemma 1 in [**G2**].

We are interested in $H^1(M^q; \mathbb{Q}/\mathbb{Z})$, the characters on $H_1(M^q)$. Define ε_q to be the endomorphism of $H_1(F)$ given by $s^q - (s-1)^q$ and $N^q \subset H_1(F; \mathbb{Q}/\mathbb{Z})$ to be the kernel of $\varepsilon_q \otimes id_{\mathbb{Q}/\mathbb{Z}}$. Gilmer [**G2**] proved that $H^1(M^q; \mathbb{Q}/\mathbb{Z})$ is isomorphic to N^q and the isomorphism can be uniquely constructed up to covering translations. So we may view the Casson-Gordon invariants $\tau(K,\ )$ as a function on N^q.

Let N^q_p denote the p-primary component of N^q. Gilmer [**G2**] proved:

THEOREM 2.17 (Gilmer). *If F is a Seifert surface for a slice knot K then there is a metabolizer Z for the isometric structure on $H_1(F)$ such that $\tau(K, N^q_p \cap (Z \otimes \mathbb{Q}/\mathbb{Z}))$ vanishes for all prime powers q and primes p.*

3.7. Gilmer's concordance group Γ^+. Gilmer [**G2**] defined a Witt type group Γ^+ and a homomorphism from the knot concordance group $\mathcal{C}$ to Γ^+. More precisely, the concordance class of a knot K maps to zero in Γ^+ if and only if it satisfies the conclusion of Theorem 2.17, namely, if F is a Seifert surface for K then there is a metabolizer Z for the isometric structure on $H_1(F)$ such that $\tau(K, N^q_p \cap (Z \otimes \mathbb{Q}/\mathbb{Z}))$ vanishes for all prime powers q and primes p. Note that a knot which maps to zero in Γ^+ is algebraically slice. Hence Levine's homomorphism $\mathcal{C} \to \mathcal{G}$ factors through Γ^+. Gilmer [**G1**] provided examples to show vanishing in Γ^+ is stronger than being algebraic slice and having Casson-Gordon's obstructions vanish.

4. Proof of Theorem 1.1

Let F_1 and F_2 be Seifert surfaces for K_1 and K_2. Then the boundary connected sum $F_1 \natural F_2$ is a Seifert surface for $K_1 \# K_2$. Let Z be a metabolizer for the isometric structure on $H_1(F_1 \natural F_2)$ as in Theorem 2.17. Then by Lemma 2.7 there are metabolizers Z_1 and Z_2 for the isometric structures on $H_1(F_1)$ and $H_1(F_2)$, respectively, with $Z = Z_1 \oplus Z_2$. Let $N = \ker \varepsilon_q \otimes id_{\mathbb{Q}/\mathbb{Z}}$ and $N_i = \ker \varepsilon_{i,q} \otimes id_{\mathbb{Q}/\mathbb{Z}}$, where ε_q and $\varepsilon_{i,q}$ are endomorphisms of H and H_i, respectively, as denoted in Subsection 3.6. Then since $\varepsilon_q = \varepsilon_{1,q} \oplus \varepsilon_{2,q}$, $N = N_1 \oplus N_2$, and $Z = Z_1 \oplus Z_2$,

$$N \cap (Z \otimes \mathbb{Q}/\mathbb{Z}) = (N_1 \cap Z_1 \otimes \mathbb{Q}/\mathbb{Z}) \oplus (N_2 \cap Z_2 \otimes \mathbb{Q}/\mathbb{Z}).$$

Let N_p and $N_{i,p}$ denote the p-primary component of N and N_i, respectively. Let $\chi_1 \in N_{1,p} \cap (Z_1 \otimes \mathbb{Q}/\mathbb{Z})$. Then $\chi = \chi_1 \oplus 0$ is an element in $N_p \cap (Z \otimes \mathbb{Q}/\mathbb{Z})$, where 0 stands for the trivial character in $N_2 \cap (Z_2 \otimes \mathbb{Q}/\mathbb{Z})$. By Theorem 2.17 and by the additivity of Casson-Gordon invariant (Theorem 2.14),

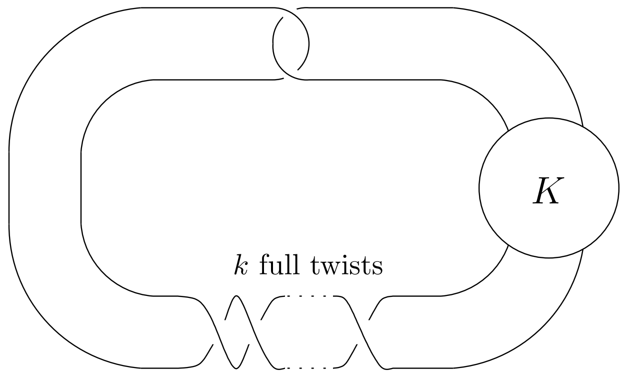
$$0 = \tau(K_1 \# K_2, \chi) = \tau(K_1, \chi_1) + \tau(K_2, 0).$$

Theorem 2.15 shows that τ is determined by the algebraic concordance class of the knot if the character is trivial. This implies that $\tau(K_2, 0) = 0$ since K_2 is algebraically slice by Theorem 2.8. Thus we have $\tau(K_1, \chi_1) = 0$. Since χ_1 was chosen arbitrary, we just found a metabolizer Z_1 for the isometric structure on $H_1(F_1)$ such that $\tau(K_1, N_{1,p} \cap (Z_1 \otimes \mathbb{Q}/\mathbb{Z}))$ vanishes for all primes p, *i.e.*, K_1 is zero in Γ^+. Similarly, K_2 is zero in Γ^+ as well. This completes proof.

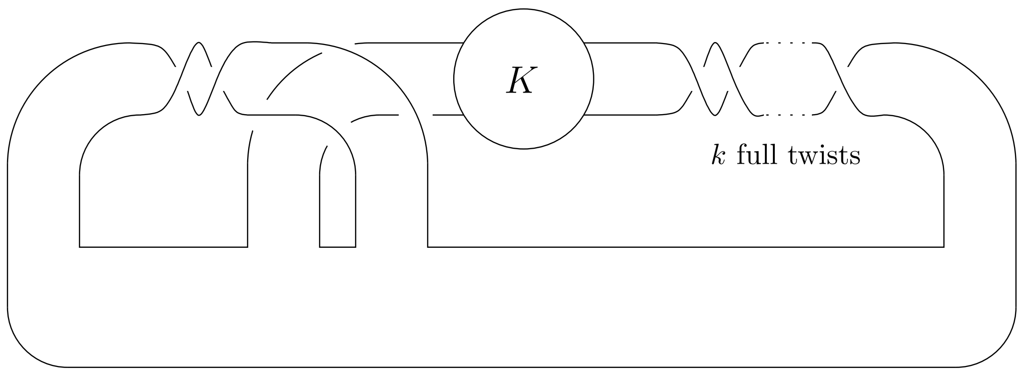
5. Twisted doubles of a knot

In this section, we define twisted doubles of a knot and show Theorem 1.2: all but finitely many twisted double of a knot has infinite order in the knot concordance group $\mathcal{C}$, in fact, in Gilmer's concordance group Γ^+.

5.1. Definition and results. Let K be a knot in the 3-sphere S^3. The *k-twisted double of the knot* K is the knot illustrated in Figure 2.1. The clasp in Figure 2.1 (a) is important. If the opposite clasp is used the results will not follow, while this case



(a)



(b)

FIGURE 2.1. The k-twisted double of a knot K

can be described as mirror images of twisted doubles. The two knots in Figure 2.1 (a) and (b) are the same. Figure 2.1 (a) is the usual description of the k-twisted double, while Figure 2.1 (b) shows its Seifert surface of genus 1 clearly. Here, k may be negative.

The following theorem is an immediate consequence of Levine [**Lev1**, Corollary 23].

THEOREM 2.18 (Levine). *The k-twisted double of a knot K is:*

(a) of infinite order in the algebraic concordance group, $\mathcal{G}$, if $k < 0$.

(b) algebraically slice if $k \geq 0$ and $4k + 1$ is a perfect square.

(c) of order 2 *in* $\mathcal{G}$ *if* $k > 0$, $4k+1$ *is not a perfect square, and every prime congruent to* 3 *mod* 4 *has even exponent in the prime power factorization of* $4k+1$.

(d) of order 4 *in* $\mathcal{G}$ *if* $k > 0$ *and some prime congruent to* 3 *mod* 4 *has odd exponent in* $4k+1$.

It follows that $D_k(K)$ has infinite order in Γ^+ if $k < 0$. We will further prove

THEOREM 2.19. *(a) For any knot* K*, the* k*-twisted double* $D_k(K)$ *has infinite order in* Γ^+ *for all but finitely many* k*. More precisely, the set* $\mathcal{K} = \{n \in \mathbb{Z} \mid D_n(K)$ *has infinite order in* $\Gamma^+\}$ *has finite complement in* $\mathbb{Z}$.

(b) If $\sigma_r(K) \geq 0$, $D_k(K)$ *has infinite order in* Γ^+ *for* $k \neq 0, 1, 2$*, and, in addition, if* $\sigma_r(K) > 0$ *for* $r = \frac{2}{9}, \frac{1}{3}, \frac{2}{5}$*, then* $D_1(K)$ *and* $D_2(K)$ *have infinite order in* Γ^+*, where* $\sigma_r(K)$ *denotes the Tristram-Levine signature of* K.

A proof of this theorem will be given later.

5.2. Casson-Gordon invariants of a genus 1 knot. We state Theorem 7 of Naik [**Nai**], which gives a formula for τ in terms of the classical signatures of knots to compute Casson-Gordon invariant of $D_k(K)$. We remark that Gilmer [**G1**] first found the formula for the 2-fold branched cover case.

We will restrict out attention to genus 1 knots. Let us assume that the Seifert surface F for the knot K is a genus 1 surface. Suppose $\{x, y\}$ be a basis of $H_1(F)$. By changing the sign of y if necessary, we can assume that the corresponding Seifert matrix is

$$A = \begin{pmatrix} a & -m \\ -(m+1) & b \end{pmatrix}.$$

Recall that N^q denote the kernel of $\varepsilon_q \otimes id_{\mathbb{Q}/\mathbb{Z}} \colon H_1(F; \mathbb{Q}/\mathbb{Z}) \to H_1(F; \mathbb{Q}/\mathbb{Z})$, where ε_q is the endomorphism of $H_1(F)$ given by $s^q - (s-1)^q$, *i.e.*, by a matrix $\left[(A - A^t)^{-1} A\right]^q - \left[(A - A^t)^{-1} A^t\right]^q$.

Let d be a power of a prime, and let s be a positive integer relatively prime to d and less than d. Suppose that $x \otimes \frac{s}{d} \in N^q$ and $d \mid a$. Let J_x be a simple closed curve on F representing x. Let m^* denote the multiplicative inverse of m (mod d).

Recall that the invariant $\tau(K, \chi)$ is an element of the Witt group $W(\mathbb{C}(t)) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $W(\mathbb{R})$ denote the Witt group of finite dimensional inner product spaces over $\mathbb{R}$. The usual signature function $\sigma : W(\mathbb{R}) \to \mathbb{Z}$ is an isomorphism. Also there is a natural map $W(\mathbb{R}) \to W(\mathbb{C}(t))$ given by tensoring with $\mathbb{C}(t)$ over $\mathbb{R}$. Composing this map with σ^{-1} tensored with $\mathbb{Q}$ gives a homomorphism $\rho \colon \mathbb{Q} \to W(\mathbb{C}(t)) \otimes_{\mathbb{Z}} \mathbb{Q}$. Recall that for each complex number ζ with $|\zeta| = 1$, there is a homomorphism $\sigma_\zeta \colon W(\mathbb{C}(t)) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ (see [**CaG**] and subsection 3.3). It is easy to see that $\sigma_1 \circ \rho$ is the identity.

For $0 < r < 1$, define $\sigma_r(K) = \sigma\left((1 - e^{2\pi i r})\,\theta + (1 - e^{-2\pi i r})\,\theta^t\right)$ as before, where θ is a Seifert form of K. This invariant $\sigma_r(K)$ is equal to Tristram-Levine signature of K at r except perhaps at finitely many points. We remark that those exceptions do not occur in our case.

The following is due to Naik [**Nai**], and the case $q = 2$ is due to Gilmer [**G1**].

Theorem 2.20 (Gilmer, Naik). *Let us assume all the notations above. Suppose q is a power of a prime. For $0 \le i \le q-1$, let $s_i \in \mathbb{Z}$ be such that $0 < s_i < d$ and $s_i \cong [m^*(m+1)]^i s \mod d$. Then $x \otimes \frac{s}{d}$ defines a character χ and*

$$\tau(K, \chi) = \rho \sum_{i=0}^{q-1} \left[\sigma_{\frac{s_i}{d}}(J_x) + \frac{2(d - s_i)s_i}{d^2} \theta(x, x) - \sigma_{\frac{i}{q}}(K) \right].$$

In particular, if $q = 2$, then $\tau(K, \chi) = \rho\left(2\sigma_{\frac{s}{d}}(J_x) + \frac{4(d-s)s}{d^2}\theta(x, x) - \sigma_{\frac{1}{2}}(K)\right)$.

5.3. $(2, 2k+1)$ **torus knot.** Let $T_{2,2k+1}$ denote the $(2, 2k+1)$ torus knot. We compute the signature σ_r of $T_{2,2k+1}$. The case $r = \frac{p}{2k+1}$, p prime, was obtained by

Tristram [**Tri**] and the general case was obtained by Litherland [**Lit1**]. Here we give a simpler proof for our case.

PROPOSITION 2.21. *$\sigma_r(T_{2,2k+1})$ is a step function:*

$$\sigma_r(T_{2,2k+1}) = -2\left[r(2k+1)+\frac{1}{2}\right]$$

for $0<r<\frac{1}{2}$ with $r \neq \frac{2l-1}{2(2k+1)}$ for any integer l, where $[z]$ denotes the largest integer which is less than or equal to z.

PROOF. A Seifert matrix of $T_{2,2k+1}$ is a $2k \times 2k$ matrix

$$B_{2k+1} = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

The Alexander polynomial is known to be $\Delta_{T_{2,2k+1}} = \frac{(t^{2(2k+1)}-1)(t-1)}{(t^2-1)(t^{2k+1}-1)}$, and it has distinct roots $e^{2\pi i \frac{2l-1}{2(2k+1)}}$ for $l=1,2,3,\ldots,k$. So, $\sigma_r(T_{2,2k+1})$ may have a jump ± 2 or 0 only at $r=\frac{2l-1}{2(2k+1)}, l=1,2,\ldots,k$, by Theorem 2.11. Note $\sigma_0=0$. To compute $\sigma_{\frac{1}{2}}$, let

$$E_e^j = \begin{pmatrix} -e & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 \end{pmatrix}$$

be a $j \times j$ matrix. It is easy to see that E_e^j is similar to the block sum of $(-e)$ and $E^{j-1}_{2-\frac{1}{e}}$. If $e>1$, then $2-\frac{1}{e}>1$. So $\sigma_{\frac{1}{2}}(T_{2,2k+1}) = \sigma(E_2^{2k}) = -2k$ by induction. Since $\sigma_0=0$ and $\sigma_{\frac{1}{2}}=-2k$, σ_r decreases by 2 at each $r=\frac{2l-1}{2(2k+1)}$. So we have $\sigma_r(T_{2,2k+1})=-2l$ if $\frac{2l-1}{2(2k+1)}<r<\frac{2l+1}{2(2k+1)}$, or $l<r(2k+1)+\frac{1}{2}<l+1$. This completes the proof. □

5.4. Estimation of Casson-Gordon invariant of $D_k(K)$. In this section we will estimate the Casson-Gordon invariant of $D_k(K)$ for $k > 0$. Recall from Theorem 2.18 that if $k > 0$, then $D_k(K)$ has finite order in the algebraic concordance group $\mathcal{G}$.

A Seifert matrix for $D_k(K)$ corresponding to the Seifert surface in Figure 2.1 is $\begin{pmatrix} -1 & 1 \\ 0 & k \end{pmatrix}$. By changing a basis to $\{x = (1,2), y = (0,1)\}$, the Seifert matrix changes to a matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^t \begin{pmatrix} -1 & 1 \\ 0 & k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4k+1 & 2k+1 \\ 2k & k \end{pmatrix}.$$

In this case, $a = 4k+1$, $m = -(2k+1)$, and $b = k$ following the notation of Theorem 2.20. Note $m^* = -2 \pmod{4k+1}$ and $m^*(m+1) = -1 \pmod{4k+1}$.

We consider the case $q = 2$. The map ε_2 is represented by the matrix

$$\left[\left(A - A^t\right)^{-1} A\right]^2 - \left[\left(A - A^t\right)^{-1} A^t\right]^2 = \begin{pmatrix} -(4k+1) & -2k \\ 2(4k+1) & 4k+1 \end{pmatrix}.$$

Let p be a prime dividing $4k+1$ and let s be an integer with $0 < s < p$. Note p is odd and $\varepsilon_2(x \otimes \frac{s}{p}) = 0$ in $H_1(F) \otimes \mathbb{Q}/\mathbb{Z}$, *i.e.*, $x \otimes \frac{s}{p}$ is in the kernel of ε_2. Note J_x, a simple closed curve on F representing $x = (1,2)$, can be represented by $K(T_{2,2k+1})$, a satellite knot of K with orbit $(2, 2k+1)$ torus knot, $T_{2,2k+1}$, as shown in Figure 2.2.

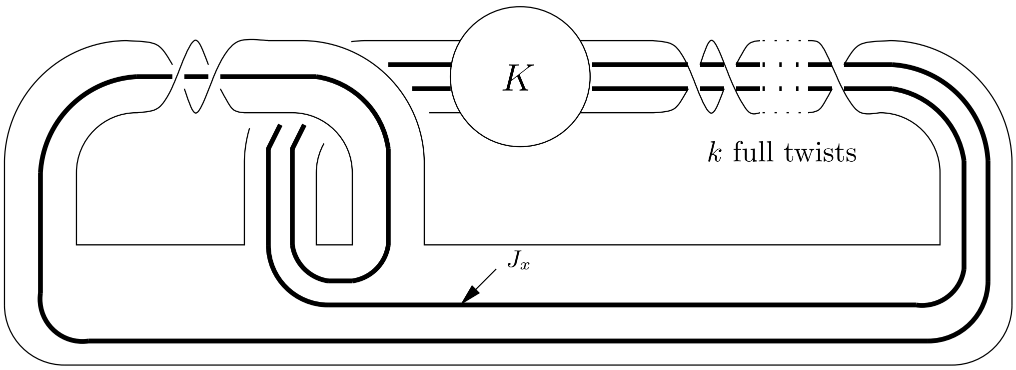
Let $\chi_{\frac{s}{p}} = x \otimes \frac{s}{p}$ be a character. Since $\sigma_{\frac{s}{p}}(K(T_{2,2k+1})) = \sigma_{\frac{2s}{p}}(K) + \sigma_{\frac{s}{p}}(T_{2,2k+1})$ by Theorem 2.12,

$$\tau(D_k(K), \chi_{\frac{s}{p}}) = \rho\left(2\sigma_{\frac{2s}{p}}(K) + 2\sigma_{\frac{s}{p}}(T_{2,2k+1}) + \frac{4(p-s)s}{p^2}(4k+1) - \sigma_{\frac{1}{2}}(D_k(K))\right).$$

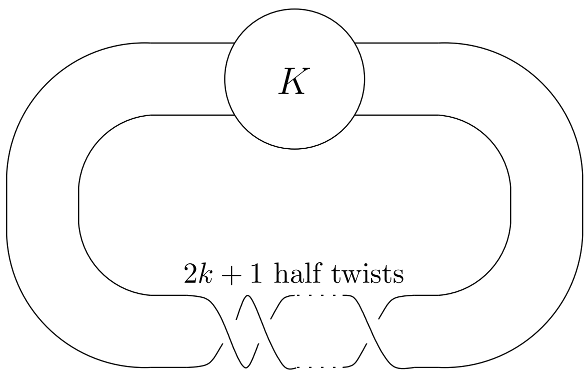
Since we are assuming that $k < 0$ and so $D_k(K)$ have finite algebraic order, and since $\sigma_{\frac{1}{2}}$ is additive under connected sums, $\sigma_{\frac{1}{2}}(D_k(K)) = 0$. Thus, we have

$$\sigma(D_k(K), \chi_{\frac{s}{p}}) = 2\sigma_{\frac{s}{p}}(K) + 2\sigma_{\frac{s}{p}}(T_{2,2k+1}) + 4\left(\frac{s}{p}\right)\left(1 - \frac{s}{p}\right)(4k+1).$$

We will show



(a) J_x in $D_k(K)$



(b) $J_x = K(T_{2,2k+1})$

FIGURE 2.2. A knot J_x that represents $x = (1, 2)$

LEMMA 2.22. *Let* $\mathrm{Min} = \min_{0<r<1} 2\sigma_r(K)$.

(a) For any $k \geq 3$,

$$\min_{\chi} \sigma(D_k(K), \chi) \geq \mathrm{Min} - \frac{4}{4k+1},$$

where χ *runs over all prime power characters.*

(b) Let s *be an integer such that* $p = 4s \pm 1$, *and let* $c = \frac{4k+1}{p}$. *Then for any constant* C_0, *there is* k_0 *such that, for any* $k \geq k_0$,

$$\sigma(D_k(K), \chi_{\frac{s}{p}}) > C_0.$$

(c) Suppose that Min ≥ 0. *If* $k \geq 3$ *then* $\sigma(D_k(K), \chi_r)$ *may have only one nonpositive value* $-\frac{4}{4k+1}$ *at* $r = \frac{1}{4k+1}$, *and*

$$\sigma(D_k(K), \chi_{\frac{s}{p}}) > \begin{cases} cs - 2 & \text{if } p = 4s+1, \\ \dfrac{c(2s-1)}{2} - 2 & \text{if } p = 4s-1. \end{cases}$$

Proof. Let $r = \frac{s}{p}$, where $0 < s < p$. Since $\sigma(D_k(K), \chi_r) = \sigma(D_k(K), \chi_{1-r})$, to compute the Casson-Gordon signature invariant σ, it suffices to compute it when $1 \leq s \leq \frac{p-1}{2}$. Then $\frac{1}{4k+1} \leq r \leq \frac{2k}{4k+1}$. For $\frac{1}{4k+1} \leq r \leq \frac{2k}{4k+1}$, let

$$f(r) := \frac{1}{2}\left(\sigma(D_k(K), \chi_r) - 2\sigma_r(K)\right) = \sigma_r(T_{2,2k+1}) + 2r(1-r)(4k+1).$$

From Proposition 2.21, we have $-2(2k+1)r - 1 \leq \sigma_r(T_{2,2k+1}) \leq -2(2k+1)r + 1$. Let

$$g(r) := -2(2k+1)r - 1 + 2r(1-r)(4k+1) = -2(4k+1)r^2 + 4kr - 1.$$

Then $g(r) \leq f(r)$. Observe that g is a quadratic polynomial in r with maximum at $r = \frac{k}{4k+1}$ and that $g\left(\frac{1}{4k+1}\right) = g\left(\frac{2k-1}{4k+1}\right) = -\frac{3}{4k+1}$, $g\left(\frac{2}{4k+1}\right) = g\left(\frac{2k-2}{4k+1}\right) = \frac{4k-9}{4k+1}$, and $g\left(\frac{2k}{4k+1}\right) = -1$.

For part (a) and the first part of (c), assume that $k \geq 3$. Then $g\left(\frac{2}{4k+1}\right) = g\left(\frac{2k-2}{4k+1}\right) = \frac{4k-9}{4k+1} > 0$, and so $f(r) \geq g(r) > 0$ if $\frac{2}{4k+1} \geq r \geq \frac{2k-2}{4k+1}$. We will compute $f(r)$ when $r = \frac{1}{4k+1}, \frac{2k-1}{4k+1}, \frac{2k}{4k+1}$. By Proposition 2.21,

$$\sigma_r(T_{2,2k+1}) = \begin{cases} -2 & \text{if } r = \frac{1}{4k+1}, \\ -2k & \text{if } r = \frac{2k-1}{4k+1}, r = \frac{2k}{4k+1}. \end{cases}$$

So,

$$f(r) = \begin{cases} -\frac{2}{4k+1} & \text{if } r = \frac{1}{4k+1}, \\ \frac{2(k-2)}{4k+1} & \text{if } r = \frac{2k-1}{4k+1}, \\ \frac{2k}{4k+1} & \text{if } r = \frac{2k}{4k+1}. \end{cases}$$

Since $k \geq 3$, $f(r)$ can be negative only when $r = \frac{1}{4k+1}$, and the value is the minimum. Thus, $\min_\chi \sigma(D_k(K), \chi) = \min_r (2\sigma_{2r}(K) + 2f(r)) \geq \text{Min} - \frac{4}{4k+1}$. This proves (a). The case Min ≥ 0 is the first part of (c).

Next, to prove part (b) and the second part of (c), we compute $f\left(\frac{s}{p}\right)$, where $p = 4s \pm 1$. Let $c = \frac{4k+1}{p}$. By Proposition 2.21 we have $\sigma_{\frac{s}{p}}(T_{2,2k+1}) = -2\left[\frac{s}{p}(2k+1) + \frac{1}{2}\right] = -2\left[\frac{cs+1}{2} + \frac{s}{2p}\right]$ since $2k+1 = \frac{cp+1}{2}$. Observe that $0 < \frac{s}{2p} = \frac{s}{8s\pm 2} \leq \frac{1}{6}$ since $s \geq 1$ and so $8s \pm 2 \geq 6s$. Since $\frac{cs+1}{2}$ is either an integer or an integer plus $\frac{1}{2}$, $\sigma_{\frac{s}{p}}(T_{2,2k+1}) = -2\left[\frac{cs+1}{2}\right]$. Then

$$\begin{aligned} f\left(\frac{s}{p}\right) &= -2\left[\frac{cs+1}{2}\right] + 2\frac{s}{p}\left(1 - \frac{s}{p}\right)cp \\ &\geq -(cs+1) + \frac{2cs(p-s)}{p} \\ &= \frac{cs(p-2s)}{p} - 1. \end{aligned}$$

If $p = 4s+1$, then $\frac{p-2s}{p} = \frac{2s+1}{4s+1} > \frac{1}{2}$. If $p = 4s-1$, then $\frac{s}{p} = \frac{s}{4s-1} > \frac{1}{4}$. Thus we have

$$\begin{aligned} f\left(\frac{s}{p}\right) &> \begin{cases} \frac{cs}{2} - 1 & \text{if } p = 4s+1, \\ \frac{c(2s-1)}{4} - 1 & \text{if } p = 4s-1. \end{cases} \\ &\geq \frac{cs}{4} - 1. \end{aligned}$$

Note $\frac{cs}{4k+1} = \frac{s}{p} = \frac{s}{4s\pm 1} \geq \frac{1}{5}$ or $cs \geq \frac{4k+1}{5}$. So if k is sufficiently large then so is $\frac{cs}{4} - 1$. This completes the proof. □

5.5. Proofs.

Proof of Theorem 2.19. If $k < 0$ then $D_k(K)$ is of infinite order by Theorem 2.18. Livingston and Naik [**LN2**] prove that all $D_k(K)$ of algebraic order 4 has infinite order in the knot concordance group. So, it remains to prove theorem for $D_k(K)$ of algebraic order 1 or 2, though proof given here works for algebraic order 4 cases as well. We remark here that in case K is the unknot, Casson and Gordon [**CaG**] showed that algebraically slice $D_k(K)$ has infinite order in $\mathcal{C}$, and

Tamulis [**Tam**] showed that $D_k(K)$ of algebraic order 2 with $4k+1$ prime has infinite order in $\mathcal{C}$.

To show that $D_k(K)$ is of infinite order in Γ^+, we show that $\#_e D_k(K)$ is not zero in Γ^+ for any integer $e > 0$. Suppose that $\#_e D_k(K)$ is zero in Γ^+ for a positive integer e. Then it satisfies the conclusion of Theorem 2.17, *i.e.*, there is a metabolizer Z for the isometric structure on $H_1(F)$ such that $\tau(\#_e D_k(K), N_p \cap (Z \otimes \mathbb{Q}/\mathbb{Z}))$ vanishes for all prime p, where F is a Seifert surface of $\#_e D_k(K)$ and N_p is the p-primary of the kernel N of $\varepsilon_2 \otimes id_{\mathbb{Q}/\mathbb{Z}}$.

Let F' be the Seifert surface for $D_k(K)$ as depicted in Figure 2.1 (b). Here we can take F as the boundary connected sum of e copies of F'. Since ε_2 is the direct sum of e copies of the map $\varepsilon_2'\colon H_1(F') \to H_1(F')$ corresponding to $D_k(K)$, N is the direct product of e copies of the subgroup $N' = \ker \varepsilon_2' \otimes id_{\mathbb{Q}/\mathbb{Z}}$ in $H_1(F') \otimes \mathbb{Q}/\mathbb{Z}$. Observe that $N' = ({\varepsilon_2'}^{-1} \otimes id_{\mathbb{Q}/\mathbb{Z}})(\mathbb{Z} \oplus \mathbb{Z})$ is generated by $\left(\frac{1}{4k+1}, 0\right) = x \otimes \frac{1}{4k+1}$, where x is a generator of $H_1(F')$ as denoted in Section 5.2. So every character in $N_p \cap (Z \otimes \mathbb{Q}/\mathbb{Z})$ is a direct sum of characters of the form $x \otimes \frac{s}{p^n}$.

Also observe that N' is isomorphic to the cyclic group $\mathbb{Z}/(4k+1)$ of order $4k+1$. So the p-primary N_p is isomorphic to $(\mathbb{Z}/p^n)^e$, where p^n is the maximal power of p dividing $4k+1$. We denote by $\bar{N}$ the group $N \cap (Z \otimes \mathbb{Q}/\mathbb{Z})$. Gilmer [**G2**, Lemma 2] showed that the order of $\bar{N}$ is the square root of the order of N. So the order of the p-primary $\bar{N}_p$ of $\bar{N}$ is the square root of the order of N_p.

Using this and the Gauss-Jordan algorithm, Livingston and Naik [**LN2**, in the proof of Theorem 1.2] showed that $\bar{N}_p$ has an element in $(\mathbb{Z}/p^n)^e \cong N_p$ with the first $e - e_0$ entries equal to p^{n-1}, where $e_0 \leq \frac{e}{2}$. Let s be an integer for which $p = 4s \pm 1$. By multiplying by s, we see that $\bar{N}_p$ has an element χ of the form $(\underbrace{x \otimes \frac{s}{p}, \ldots, x \otimes \frac{s}{p}}_{e-e_0}, x \otimes \frac{s_1}{p}, \ldots, x \otimes \frac{s_{e_0}}{p})$, where s_i can be any integer. We will compute the Casson-Gordon invariant for χ. By the additivity of τ, we have

$$\sigma(\#_e D_k(K),\chi) = (e-e_0)\sigma\left(D_k(K),\chi_{\frac{s}{p}}\right) + \sum_{i=1}^{e_0}\sigma\left(D_k(K),\chi_{\frac{s_i}{p}}\right).$$

Let Min be the minimum of $2\sigma_r(K)$ as denoted previously. By Lemma 2.22 (a) there is $k_0 > 0$ such that $\sigma(D_k(K),\chi_{\frac{s}{p}}) > -\text{Min}+1 > -\text{Min}+\frac{4}{4k+1}$ for any $k \geq k_0$. By Lemma 2.22 (b),

$$\sigma(\#_e D_k(K),\chi) > (e-e_0)\left(-\text{Min}+\frac{4}{4k+1}\right) + e_0\left(\text{Min}-\frac{4}{4k+1}\right) \geq 0$$

since $e-e_0 \geq e_0$. This implies that $\#_e D_k(K)$ is not zero in Γ^+. This contradicts the assumption that $\#_e D_k(K)$ is zero in Γ^+. Therefore, we see that $D_k(K)$ is of infinite order in Γ^+ for sufficiently large k. This proves part (a) of theorem.

Next, assume Min ≥ 0 and $k \geq 3$. Let χ denote the character as given above again. If $4k+1$ is a composite number, *i.e.*, $\frac{4k+1}{p} > 1$, then $\sigma(D_k(K),\chi) > 0$ by Lemma 2.22 (c) since any $\frac{s}{p}, \frac{s_i}{p}$ cannot be $\frac{1}{4k+1}$. Assume $4k+1 = p$. Then $c = 1$ and $s = k$, where c and s are those in Lemma 2.22, and so by Lemma 2.22 (c) $\sigma(D_k(K),\chi_{\frac{k}{p}}) > k-2 > \frac{4}{4k+1}$. We have

$$\sigma(\#_e D_k(K),\chi) > (e-e_0)\frac{4}{4k+1} + e_0\frac{-4}{4k+1} \geq 0.$$

So $D_k(K)$ has infinite order in Γ^+, which proves the first part of (b).

For $k = 1, 2$, a direct computation shows:

k	1		2			
r	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$
$\frac{1}{2}\sigma(D_k(K),\chi_r) - \sigma_{2r}(K)$	$-\frac{2}{5}$	$\frac{2}{5}$	$-\frac{2}{9}$	$\frac{10}{9}$	0	$\frac{4}{9}$

So if $\sigma_{2r}(K) > 0$ for $r = \frac{1}{5}, \frac{1}{9}, \frac{3}{9}$, then $\sigma(D_k(K),\chi_{\frac{s}{4k+1}})$ are all positive for $k = 1, 2$. Note that $\sigma_{\frac{6}{9}}(K) = \sigma_{\frac{1}{3}}(K)$. Therefore $D_1(K)$ and $D_2(K)$ are of infinite order in Γ^+ if $\sigma_r(K) > 0$ for $r = \frac{2}{9}, \frac{1}{3}, \frac{2}{5}$. This completes proof. ∎

Proof of Theorem 1.2. The Alexander polynomial of $D_k(K)$ is $\Delta_k = -kt^2 + (2k+1)t - k$. We will see that all Δ_k ($k \neq 0$) are coprime in $\mathbb{Q}[t^{\pm 1}]$. Let $k \neq l$ and let g be the greatest common divisor of Δ_k and Δ_l. Then g divides $l\Delta_k - k\Delta_l = (l-k)t$. Since t is not a factor of Δ_k for any $k \neq 0$, g is a constant number. Hence Δ_k and Δ_l are relatively prime in $\mathbb{Q}[t^{\pm 1}]$ for any distinct pair l, k. Every $D_k(K)$, $k \neq 0$, has a non-singular Seifert form since its Seifert matrix is $\begin{pmatrix} -1 & 1 \\ 0 & k \end{pmatrix}$. Therefore, twisted doubles of a knot satisfy the condition of Theorem 1.1.

Let $\mathcal{K}$ be the index set of k's for which $D_k(K)$ has infinite order in Γ^+. Then by Theorem 2.19 the set $\mathcal{K}$ contains all but finitely many integers. Suppose that there are distinct $k_i \in \mathcal{K}$ such that $e_1 D_{k_1}(K) \# e_2 D_{k_2}(K) \# \cdots \# e_n D_{k_n}(K)$ is zero in Γ^+, where $e_i D_{k_i}(K)$ is an abbreviation of $\#_{e_i} D_{k_i}(K)$. Since $\Delta_{e_i D_{k_i}(K)}(t) = \left(\Delta_{D_{k_i}(K)}(t)\right)^{e_i}$, all $\Delta_{e_i D_{k_i}(K)}(t)$ are coprime. By applying Theorem 1.1 inductively, we see that each $e_i D_{k_i}(K)$ is zero in Γ^+, and hence $e_i = 0$ since $i \in \mathcal{K}$. This completes the proof. □

Proof of Corollary 1.3. Part (a) follows immediately from Theorem 1.2 (b) since $\sigma_r(U) = 0$ for any r if U is the unknot. For part (b), the same argument given for the proof of Proposition 2.21 can show that the torus knots $T_{-2,2k+1}$ satisfies the condition of Theorem 1.2 (c) if $k \geq 2$. □

6. Another application of Theorem 1.1

In this section, we construct examples of linearly independent algebraically slice knots with homology groups of the same order on all prime power fold branched covers. To do it, we need some algebraic background.

The *resultant* of two non-constant integral polynomials $f(t)$ and $g(t)$ is defined as follows: We may factor completely the polynomials f and g in some overring of the integers as: $f(t) = a\prod_{i=1}^{n}(t - \alpha_i)$ and $g(t) = b\prod_{j=1}^{m}(t - \beta_j)$. Then the *resultant* of f and g, denoted $R(f,g)$, is $R(f,g) = a^m b^n \prod_{i=1}^n \prod_{j=1}^m (\alpha_i - \beta_j)$. Note that $R(f,g) = b\prod_{j=1}^m f(\beta_j)$, where $g(t) = b\prod_{j=1}^m (t-\beta_j)$.

It is known (see [**Web**] and Theorem 2.1) that the order of the homology of an n-fold cyclic cover of S^3, branched over a knot K, is the absolute value of the resultant $|R(t^n-1, \Delta_K(t))|$ of $t^n - 1$ and of $\Delta_K(t)$, the Alexander polynomial of the knot.

By $\phi_m(t)$ we denote the m-th cyclotomic polynomial. By $\varphi(m)$ we denote the Euler function, *i.e.*, the number of positive integers which are less than m and relatively prime to m. Let $m = p^a q^b$, where p, q are distinct primes and a, b are positive integers. If n is a prime power, Proposition 3.4 of Stoltzfus [**Sto1**, p. 28] states that

$$R(t^n-1, \phi_m(t)) = \begin{cases} q^{\varphi(p^a)} & \text{if } n = p^i,\ i \geq a \\ p^{\varphi(q^b)} & \text{if } n = q^j,\ j \geq b \\ 1 & \text{otherwise.} \end{cases}$$

It is immediate from the definition of resultant that $R(t^n-1, fg) = R(t^n-1, f)R(t^n-1, g)$. Using this, we have that

$$R(t^n-1, \phi_{p^{a_1}q^{b_1}}(t)\phi_{p^{a_2}q^{b_2}}(t)) = R(t^n-1, \phi_{p^{a_1}q^{b_2}}(t)\phi_{p^{a_2}q^{b_1}}(t))$$

In particular, let a, b, c be positive integers with $a > 1$, $b < c$. Then we have

$$\begin{aligned} R(t^n-1, \phi_{2^a p^b}(t)\phi_{2p^c}(t)) &= R(t^n-1, \phi_{2^a p^c}(t)\phi_{2p^b}(t)) \\ &= \begin{cases} p & \text{if } n = 2^i,\ 1 \leq i < a \\ p^{\varphi(2^a)+1} & \text{if } n = 2^j,\ j \geq a \\ 2^{\varphi(p^b)} & \text{if } n = p^i,\ b \geq i < c \\ 2^{\varphi(p^b)+\varphi(p^c)} & \text{if } n = p^i,\ i \geq c \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $\phi_m(1) = 1$ if m is not a power of prime and since $\phi_m(t) = t^{\varphi(m)}\phi_m(t^{-1})$, by Proposition 14.2 of Levine [**Lev2**, p. 236], there is a Seifert matrix A whose

Alexander polynomial is $\phi_m(t)$. So there are knots K_1 and K_2 in S^3 such that $\Delta_{K_1}(t) = \phi_{2^a p^b}(t)\phi_{2p^c}(t)$ and $\Delta_{K_2}(t) = \phi_{2^a p^c}(t)\phi_{2p^b}(t)$.

The order of the homology of the 2-fold cover $M^2_{K_i}$ of S^3, branched over $K_i, i = 1, 2$, is the prime $p = |R(x^2 - 1, \Delta_{K_i}(t))|$. Let F_i be a Seifert surface of K_i and let $\{a_{i,1}, \ldots, a_{i,2g}\}$ be a basis for $H_1(F_i)$. As in [**Rol**, p. 209], let $\{\alpha_{i,1}, \ldots, \alpha_{i,2g}\}$ denote the basis for $H_1(S^3 - F_i)$ such that $lk(a_{i,k}, \alpha_{i,j}) = \delta^k_j$. Let $\alpha^0_{i,1}, \ldots, \alpha^0_{i,2g}, \alpha^1_{i,1}, \ldots, \alpha^1_{i,2g}$ be the lifts of $\alpha_{i,1}, \ldots, \alpha_{i,2g}$ in $M^2_{K_i}$. It is known that $\alpha^0_{i,1}, \ldots, \alpha^0_{i,2g}$ generates $H_1(M^2_{K_i})$. Since $H_1(M^2_{K_i}) = \mathbb{Z}/(p)$ is cyclic of prime order, we may assume that $\alpha^0_{i,1}$ generates $H_1(M^2_{K_i})$.

Let C be a knot with $\sigma_{\frac{s}{p}}(C) > \max_\chi\{|\sigma_1\tau(K_i, \chi)|\}$ for all $s = 1, \ldots, p-1$, where χ runs over all the characters on $H_1(M^2_{K_i})$. Note that such a knot exists, for example, the torus knot $T_{-2,2k+1}$ for a sufficiently large k as shown in the proof of Corollary 1.3. We consider a satellite knot $C(K_i)$ with orbit K_i, axis $\alpha_{i,1}$, and a satellite knot $C(-K_i)$ with orbit $-K_i$, axis $-\alpha_{i,1}$, where $\alpha_{i,1}$ is reused to denote the unknot representing $\alpha_{i,1}$. Observe that both satellite knots have winding number 0 and the Seifert surface F_i of K_i lies in $S^3 - \alpha_{i,1}$. The satellite knots $C(K_i)$ and $C(-K_i)$ have the same Seifert matrices as those of K_i and $-K_i$, respectively, and hence $H_1(M^2_{\pm K_i}) = H_1(M^2_{C(\pm K_i)})$.

PROPOSITION 2.23. *The connected sum $E_i = C(K_i)\#C(-K_i)$ has an infinite order in Γ^+.*

PROOF. Suppose not, *i.e.*, there is an integer $e > 0$ such that eE_i is zero in Γ^+, where $eE_i = \#_e Ei$. Then eE_i satisfies the conclusion of Theorem 2.17. Let $\chi : H_1(M^2_{E_i}) \to \mathbb{Q}/\mathbb{Z}$ be a character. Since $H_1(M^2_{eE_i}) = \oplus_e \Big(H_1(M^2_{C(K_i)}) \oplus H_1(M^2_{C(-K_i)})\Big)$ $= \oplus_e \big(H_1(M^2_{K_i}) \oplus H_1(M^2_{-K_i})\big)$, χ can be identified with $\oplus^e_{j=1}\left(\chi^1_j \oplus \chi^2_j\right)$, where χ^1_j and χ^2_j are characters on $H_1(M^2_{C(K_i)}) = H_1(M^2_{K_i})$ and $H_1(M^2_{C(-K_i)}) = H_1(M^2_{-K_1})$, respectively. As shown in the proof of Theorem 2.19, we may assume that there is a character χ for any metabolizer in Theorem 2.17 such that at least half of χ_j^1 and χ_j^2 are nontrivial. Then, by the additivity of Casson-Gordon invariant and by Corollary 2 of Litherland [**Lit2**, p. 338] ($n = 2$ case),

$$\begin{aligned}\sigma(eE_i,\chi) &= \sum_{j=1}^{e}\big(\sigma(C(K_i),\chi_j^1)+\sigma(C(-K_i),\chi_j^2)\big)\\ &= \sum_{j=1}^{e}\left(\sigma(K_i,\chi_j^1)+\sigma(-K_i,\chi_j^2)+\sum_{l=0}^{1}\Big(\sigma_{\chi_j^1(\alpha_{i,1}^l)}(C)+\sigma_{\chi_j^2(\alpha_{i,1}^l)}(C)\Big)\right)\\ &> \sum_{j=1}^{e}\big(\sigma(K_i,\chi_j^1)+\sigma(-K_i,\chi_j^2)\big)+2e\max\{|\sigma(K_i,\cdot)|\} \quad\geq\quad 0.\end{aligned}$$

This is a contradiction. So the knot E_i has an infinite order in Γ^+. □

Since two distinct cyclotomic polynomials have no common roots, they are all coprime, and so are $\Delta_{E_1}(t)$ and $\Delta_{E_2}(t)$. By Proposition 2.23 and Theorem 1.1, we have that E_1 and E_2 are linearly independent in Γ^+.

COROLLARY 2.24. *There are linearly independent algebraically slice knots with homology groups of the same order on all prime power fold branched covers.*

Note that they are algebraically slice and, as proven previous using resultant, they have homology groups of the same order on all prime power fold branched covers. Therefore, they provide an example that our approach using the splitting associated to the polynomial is definitely required to show they are linearly independent.

We remark that all the calculations in this section can be carried out in a general case that Alexander polynomials are $\phi_{p^{a_1}q^{b_1}}(t)\phi_{p^{a_2}q^{b_2}}(t)$ and $\phi_{p^{a_1}q^{b_2}}(t)\phi_{p^{a_2}q^{b_1}}(t)$.

CHAPTER 3

Invertible knot concordances and prime knots

Kirby and Lickorish [**KL**] showed that every knot in S^3 is concordant to a prime knot, equivalently, every concordance class contains a prime knot. Generalizations appear in [**Liv, My1, My2, Som**]. Sumners [**Sum**] introduced the notion of invertible concordance. It has been proved by Nakanishi [**Nak**, Theorem 3.5] that the Kirby and Lickorish's result can be strengthened: Every knot in S^3 is invertibly concordant to a prime knot. We give a new and simple proof of this theorem.

Corresponding to invertible concordance there is a group, the *double concordance group*, studied in [**Lev3, Rub1, Sto2**]. A consequence is that every double concordance class contains a prime knot.

Throughout this chapter we will work in the category of smooth oriented manifolds and pairs.

1. Definitions and basic results

Let I denote the interval $[0,1]$. A *link* of n components, L, is a smooth pair (S^3, l) where l is a smooth oriented submanifold of S^3 diffeomorphic to n disjoint copies of S^1. A *knot* K is a link of one component. Two links, L_1 and L_2, each of n components, are called *concordant* if there exists a proper smooth oriented submanifold w of $S^3 \times I$, with $\partial w = (l_1 \times 0 \cup (-l_2) \times 1)$ and w diffeomorphic to n disjoint copies of $S^1 \times I$. Note that this definition is a generalization of the alternative definition of knot concordance given in the previous chapter. Since a connected sum of links is not well-defined, link concordance is not defined via slice links and a notion of link concordance group under connected sum is inaccessible.

Let $(W; L_1, L_2)$ be $(S^3 \times I, w)$ the concordance between L_1 and L_2. If $(W_1; L_1, L_2)$ and $(W_2; L_2, L_3)$ are two concordances with a common boundary component (oriented oppositely) we can then paste W_2 to W_1 along L_2 to get $(W_1 \cup W_2; L_1, L_3)$.

A concordance $(W; L_1, L_2)$ is said to be *invertible at* L_2 if there is a concordance $(W'; L_2, L_1)$ such that $(W \cup W'; L_1, L_1)$ is diffeomorphic to $(L_1 \times I; L_1, L_1)$, the product concordance of L_1. Given the above situation, we say that L_1 *is invertibly concordant to* L_2 or L_2 *splits* $L_1 \times I$. In the same manner, concordance and invertible concordance can be defined for knots and links in the solid torus $S^1 \times D^2$.

A submanifold N with boundary is said to be *proper* in a manifold M if $\partial N = N \cap \partial M$. Let B^3 denote the standard closed 3-ball $\{x \in \mathbb{R}^3 \mid |x| \leq 1\}$. An *$n$-tangle* T is a smooth pair (B^3, λ) where λ is a proper embedding of n disjoint copies of the interval I into B^3. Throughout this chapter an embedding means either the map or the image. Let U_n denote a trivial n-tangle, *i.e.*, U_n consists of n unlinked unknotted arcs. For example, U_1 is the unknotted standard ball pair (B^3, I). For $n = 2$, see Figure 3.1.

Concordances and invertible concordances between tangles can be defined in a similar way as for links. However, the boundary of the 3-ball B^3 is required to be fixed at each stage of concordance. More precisely, let $I_1, \ldots, I_n$, denote n disjoint copies of the interval I. Two n-tangles, $T_0 = (B^3, \lambda_0)$ and $T_1 = (B^3, \lambda_1)$, are *concordant* if there is a proper smooth embedding τ of $(\cup_{i=1}^n I_i) \times I$ into $B^3 \times I$, with $\tau(\cup_{i=1}^n I_i \times \epsilon) = \lambda_\epsilon$ $(\epsilon = 0, 1)$ and $\tau(\epsilon_i \times I) = \tau(\epsilon_i \times 0) \times I$ for each $i = 1, \ldots, n$, and $\epsilon_i = 0, 1$ in I_i. Let $(V; T_1, T_2)$ denote $(B^3 \times I, \tau)$ the concordance between T_1 and T_2. If $(V; T_1, T_2)$ and $(V'; T_2, T_3)$ are two concordances, we can then paste V' to V along T_2 to get a concordance $(V \cup V'; T_1, T_3)$. A concordance $(V; T_1, T_2)$ is *invertible at* T_2 if there is a concordance $(V'; T_2, T_1)$ such that $(V \cup V'; T_1, T_1)$ is diffeomorphic to $(T_1 \times I; T_1, T_1)$ by a diffeomorphism φ with $\varphi(\tau) = \lambda_1 \times I$, where τ is the embedding of n disjoint copies of $I \times I$ into $B^3 \times I$ defining the concordance $(V \cup V'; T_1, T_1)$ and λ_1 is the embedding of n disjoint copies of I into B^3 defining the tangle T_1.

A knot is called *doubly null concordant* or *doubly slice* if it splits $U \times I$ for the unknot U. Alternatively, a doubly null concordant knot is the slice of some unknotted 2-sphere in S^4. Two knots K_1 and K_2 are said to be *doubly concordant* if $K_1 \# J_1$ is isotopic to $K_2 \# J_2$ for some doubly null concordant knots J_1 and J_2.

The following theorem is due to Zeeman.

THEOREM 3.1. [**Z**] *Every 1-twist-spun knot is unknotted.*

The following corollary was first proved by Stallings and follows readily from Theorem 3.1. (One cross-section of the 1-twist-spin of K yields $K\#(-K)$. For details, see [**Sum**].)

COROLLARY 3.2. *$K\#(-K)$ is doubly null concordant for every knot K.*

COROLLARY 3.3. *If $K_1\#(-K_2)$ is doubly null concordant then K_1 and K_2 are doubly concordant.*

PROOF. Take $J_1 = K_2\#(-K_2)$ and $J_2 = K_1\#(-K_2)$ in the definition of double concordance. □

REMARK 3.4. Recall that knots K_1 and K_2 are concordant if $K_1\#(-K_2)$ is slice. However, a definition of double concordance more along the lines of concordance is as of yet inaccessible. The difficulty is that it is unknown whether the following is true: If knots K and $K\#J$ are doubly null concordant, then J is doubly null concordant.

There is a relation between invertible concordance and double concordance.

PROPOSITION 3.5. *If* K_1 *is invertibly concordant to* K_2 *then* $K_1\#(-K_2)$ *is doubly null concordant.*

PROOF. There is a copy of $S^3 \times I$ in S^4 intersecting the 1-twist-spin of K_1 in $K_1\#(-K_1) \times I$. Since K_2 splits $K_1 \times I$, there is an invertible concordance from $K_1\#(-K_1)$ to $K_1\#(-K_2)$. Hence $K_1\#(-K_1) \times I$ is split by $K_1\#(-K_2)$ and the result follows. □

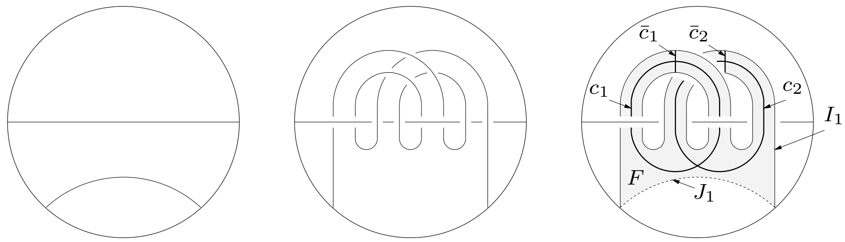
2. Invertible concordances and prime knots

Kirby and Lickorish [**KL**] proved that any knot in S^3 is concordant to a prime knot. Livingston [**Liv**] gave a different proof of this result using satellite knots. In this section we modify Livingston's approach to prove Theorem 1.4. In the rest of this chapter the orbit of a satellite knot will be considered as a knot in the solid torus $S^1 \times D^2$, the complement of the axis in S^3.

We set up some notation. By a *splitting*-S^2, S, for a knot K (in S^3 or $S^1 \times D^2$) we denote an embedded 2-sphere, S, intersecting K in exactly 2 points. A knot in either S^3 or $S^1 \times D^2$ is *prime* if every splitting-S^2 bounds some 3-ball, B, with $(B, B \cap K)$ a trivial pair. The *wrapping number* of K is the minimum number of intersections of K with a disk D in $S^1 \times D^2$ with $\partial D =$ meridian.

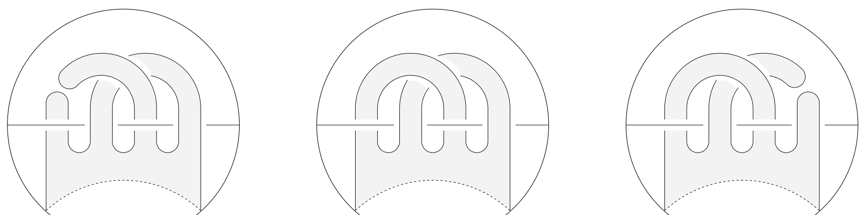
In what follows we will consider $S^1 \times D^2$ embedded in S^3 in a standard way. Hence any knot K in $S^1 \times D^2$ gives rise to a knot K^* in S^3. The following theorem is due to Livingston.

THEOREM 3.6. [**Liv**] *Let* K_1 *be a knot in* $S^1 \times D^2$ *such that* K_1^* *is the unknot in* S^3*. Then* K_1 *is prime in* $S^1 \times D^2$*. Moreover, if* K_1 *has wrapping number* > 1 *and* K_2 *is any nontrivial knot in* S^3*, then the satellite knot* $K_2(K_1)$ *of* K_2 *with orbit* K_1 *is prime in* S^3*.*



(a) U_2 (b) T (c)

FIGURE 3.1



$B^3 \times \frac{1}{4}$ $B^3 \times \frac{1}{2}$ $B^3 \times \frac{3}{4}$

FIGURE 3.2

This theorem suggests that, to prove Theorem 1.4, we only need to find a knot K_1 in $S^1 \times D^2$ with K_1^* the unknot in S^3 and an invertible concordance between the core C of $S^1 \times D^2$ and the knot K_1 in $S^1 \times D^2$. We will show that there is an invertible concordance between the tangles U_2 and T in Figure 3.1. We remark here that Ruberman in [**Rub2**] used the tangle T to prove that any closed orientable 3-manifold is invertibly homology cobordant to a hyperbolic 3-manifold.

LEMMA 3.7. *The 2-tangle T in Figure 3.1(b) splits $U_2 \times I$.*

PROOF. Let I_1 be a copy of the non-straight arc of T in the 3-ball B^3 and let J_1 be a copy of the non-straight arc of U_2 in B^3 as shown in Figure 3.1(c). The closed curve $J_1 \cup I_1$ bounds an obvious punctured torus F that is the shaded region in Figure 3.1(c). Consider F as the plumbing of two $S^1 \times I$. Let $c_i,\ i = 1, 2$, be the cores of the two $S^1 \times I$ of F and let $\bar{c}_i,\ i = 1, 2$, be disjoint proper line segments in F intersecting with c_i exactly once, respectively. See Figure 3.1(c).

To construct an invertible concordance, we will construct two concordances and then paste them together. First, note that pinching I_1 along $\bar{c}_1$ transforms T into the tangle U_2 with an unlinked unknotted circle inside which is isotopic to the circle c_2. Capping off this circle we have a concordance $(V_1'; T, U_2)$. The tangle $B^3 \times \frac{1}{4}$ in Figure 3.2 represents a slice of this concordance before capping off the circle. In the similar way, pinching I_1 along $\bar{c}_2$ and capping off the unknot gives us another concordance $(V_2; T, U_2)$. Let $(V_1; U_2, T)$ denote the concordance $(V_1'; T, U_2)$ with reversed orientation. We can then paste V_1 to V_2 along T to get a concordance $(V_1 \cup V_2; U_2, U_2)$, which will be proved to be isotopic to the product concordance $U_2 \times I$. A few cross-sections of concordance $V_1 \cup V_2$ are drawn in Figure 3.2.

Let τ denote the embedding of two disjoint copies of $I \times I$ into $V_1 \cup V_2$ as in the definition of concordance in Section 1. It is obvious from Figure 3.2 that there is a 3-manifold M (the union of shaded regions) in $V_1 \cup V_2$ bounded by τ and $J_1 \times I$, whose intersection with U_2 at each end of the concordance is the arc J_1 and whose cross-section in the middle is the punctured torus F. This 3-manifold M can be considered as the union of three submanifolds: the product $F \times I$ and two 3-dimensional 2-handles $D^2 \times I$. One $D^2 \times I$ is glued to $F \times I$ along a regular neighborhood of c_2, which corresponds to capping off the circle isotopic to c_2 as we constructed the concordance V_1'. The other $D^2 \times I$ is glued along a regular neighborhood of c_1, which corresponds to capping off the circle isotopic to c_1 as we constructed the concordance V_2. Since $F \times I$ is a 3-dimensional handlebody with 2 handles with cores c_1 and c_2, M is the manifold that results by adding two 2-handles to a genus 2 solid handlebody along the cores of the 1-handles, in this case yielding B^3. Moreover, M does not intersect the other straight arc of T at any stage. Using this 3-ball M, we can isotop τ to

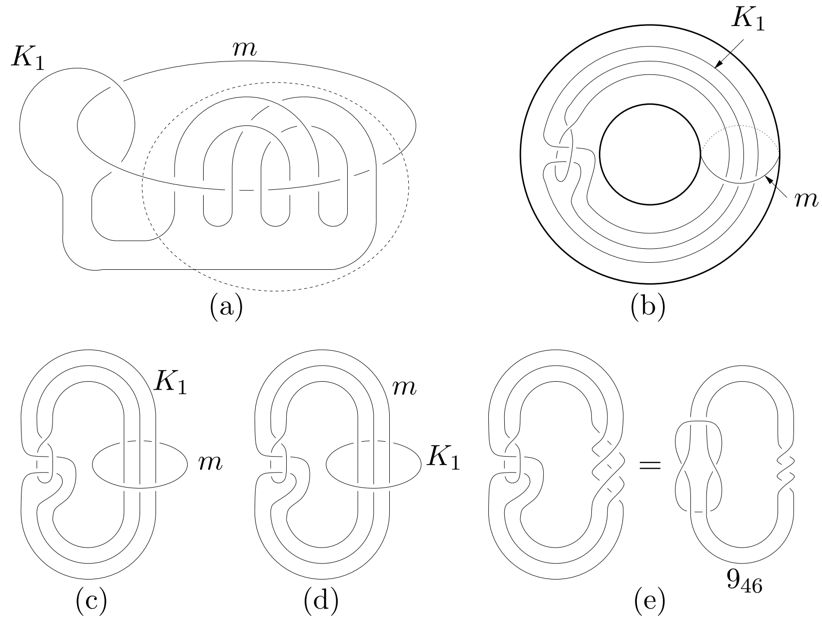


FIGURE 3.3

$J_1 \times I$ in a regular neighborhood of M not disturbing the other arc and ∂B^3. This completes the proof. □

PROPOSITION 3.8. *The knot K_1 in Figure 3.3(b) splits $C \times I$, where C is the core of $S^1 \times D^2$.*

PROOF. Consider $S^1 \times D^2$ as the complement of the unknot m in S^3. The knot K_1 in Figure 3.3(b) is isotopic to K_1 in Figure 3.3(a). It is obvious from Figure 3.3(a) that $K_1 \cup m$ is the link in S^3 formed by replacing a trivial 2-tangle in Hopf link with T (dotted circle in Figure 3.3(a)). The proposition follows from Lemma 3.7. □

We are ready to prove Theorem 1.4.

PROOF OF THEOREM 1.4. Let K be a knot in S^3. If K is trivial it is prime itself. Suppose that K is nontrivial. Let K' be K_1 satellite of K where K_1 is the knot in $S^1 \times D^2$ in Figure 3.3(b). By Proposition 3.8, K' splits $K \times I$. We only need to show K' is prime. Since K_1^* is the unknot in S^3, K_1 is prime by Theorem 3.6 and to complete proof it remains to show its wrapping number > 1. Its winding number is 1, hence its wrapping number is at least one. It is easy to see that the only prime knot in $S^1 \times D^2$ with wrapping number 1 is the core. So, if K_1 had wrapping number 1, then it is isotopic to the core of $S^1 \times D^2$. The -1 surgery on the meridian curve m in S^3 should make K_1^* unchanged, *i.e.*, unknotted. However, the knot in Figure 3.3(e), the result of K_1^* after -1 surgery along m, is 9_{46} and hence knotted. Therefore the wrapping number is > 1. □

COROLLARY 3.9. *Any knot is doubly concordant to a prime knot.*

REMARK 3.10. The K_1 satellite of K has the same Alexander polynomial as that of K. Seifert [**Sei**] proved that the Alexander polynomial of the satellite of K with orbit K_1 is $\Delta_{K_1^*}(t)\Delta_K(t^w)$ if w is the winding number of K_1 in $S^1 \times D^2$. In our case, w is 1 and K_1^* is the unknot.

In [**Liv**], Livingston also proved that every 3-manifold is homology cobordant to an irreducible 3-manifold. Two 3-manifolds M_1 and M_2 are *homology cobordant* if there is a 4-manifold W with $\partial W = M_1 \cup M_2$ and the map of $H_*(M_i;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ an isomorphism. Invertible homology cobordisms can be defined in the same way as in the knot concordance case. A 3-manifold M is *irreducible* if every embedded S^2 in M bounds an embedded B^3.

REMARK 3.11. In spirit of [**Liv**], we have a simple proof that every 3-manifold is invertibly homology cobordant to an irreducible 3-manifold. To prove this, we only need to slightly modify the proof of Theorem 3.2 in [**Liv**] by using K_1 in Figure 3.3(b). The -1 surgery on K_1 makes the meridian m the knot 9_{46}.

This remark is, in fact, a corollary of Ruberman's Theorem 2.6 in [**Rub2**] that reads: for every closed orientable 3-manifold N, there is a hyperbolic 3-manifold M, and an invertible homology cobordism from M to N. The remark follows since a hyperbolic 3-manifold is irreducible.

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