Nonlinear Inverse Problems in Imaging

Jin Keun Seo and Eung Je Woo
**General Notation**

\( \mathbb{R} \) : the set of real numbers.

\( \mathbb{R}^n \) : \( n \)-dimensional Euclidian space.

\( \mathbb{C} \) : the set of complex numbers.

\( i = \sqrt{-1} \).

\( \mathbb{N}_0 = \{0, 1, 2, \cdots\} \) : the set of nonnegative integers.

\( \mathbb{N} = \{1, 2, \cdots\} \) : the set of positive integers.

\( \mathbf{r} = (x, y, z) \) : position.

\( B_r(\mathbf{a}) \) : the ball with radius \( r \) at the center \( \mathbf{a} \) and \( B_r = B_r(0) \).

\( \mathbf{e}_j \) : \( j \)-th unit vector in \( \mathbb{R}^n \); for example, \( \mathbf{e}_2 = (0, 1, 0, \cdots, 0) \).

\( \forall \) := for all

\( \exists \) := there exist

\( \because \) := because

\( \therefore \) := therefore

**Electromagnetism**

\( \mathbf{E} \) : electric field intensity

\( \mathbf{B} \) : magnetic flux density

\( \mathbf{J} \) : current density

\( \mathbf{D} \) : electric flux density

\( \sigma \) : electrical conductivity

\( \epsilon \) : electrical permittivity

\( \gamma = \sigma + i\omega\epsilon \) : admittivity

\( u \) : voltage (electrical potential)

**Notations for domains and vector space**

\( \Omega \) : a domain in \( \mathbb{R}^n \).

\( \partial\Omega \) : the boundary of the domain \( \Omega \).

\( \mathbf{n} \) : the unit outward normal vector to the boundary .

\( C(\Omega) \) : the set of all continuous functions in \( \Omega \).

\( C^k(\Omega) \) : the set of continuously \( k \)-th differentiable functions defined in the domain \( \Omega \).
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Electrical Impedance Tomography

Electrical impedance tomography (EIT) produces cross-sectional images of an admittivity distribution inside an electrically conducting object. It has a wide range of applications in biomedicine, geophysics, non-destructive testings and so on. Considering the fact that structural imaging modalities such as X-ray CT and MRI provide images with a superior spatial resolution to EIT, the primary goal of biomedical EIT is to supply functional diagnostic information of organs with a high temporal resolution. It may classify pathological conditions of biological tissues, where other methods do not provide enough contrast to do that. Following a brief introduction to EIT, we summarize bioimpedance measurement methods, which an EIT system is based on to acquire data for image reconstructions. Its forward problem is introduced in a context of a practically feasible measurement setting. Modeling of the forward problem and sensitivity analysis will be the key to understand and design an inversion method. Three kinds of EIT inverse problems including static imaging, time-difference imaging and frequency-difference imaging will be described.

1.1 Introduction

Material properties of electrical conductivity and permittivity form image contrast in EIT. As described in section xxx, conductivity ($\sigma$) and permittivity ($\epsilon$) values of a biological tissue are determined by its ion concentrations in extra- and intra-cellular fluids, cellular structure and density, molecular compositions, membrane characteristics, and other factors. In the frequency range of a few Hz to MHz, numerous experimental findings indicate that different biological tissues have different electrical properties and their values are influenced by physiological and pathological conditions. In biomedical applications of EIT, we deal with the admittivity $\gamma$ where $\gamma = \sigma + i\omega\epsilon$ and the angular frequency $\omega = 2\pi f$ in rad/s with the frequency $f$ in Hz. For most biological tissues, we may assume that $\gamma \approx \sigma$ at low frequencies below 1 kHz. With abundant membraneous structures in an organism, the $\omega\epsilon$ term is not negligible beyond 1 kHz and we should deal with the admittivity $\gamma = \sigma + i\omega\epsilon$ in general at high frequencies.

We consider an electrically conducting object such as the human body with its internal admittivity distribution $\gamma(r)$ as a function of the position $r = (x, y, z)$. To probe the object with the intention of non-invasively sensing $\gamma$, we inject current through electrodes attached on its surface. This induces internal current density and voltage distributions that are determined by the admittivity distribution, object geometry and electrode configuration. In the frequency range up to a few MHz, we may adopt the elliptic PDE introduced in section...
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xxx to describe the interrelations among the injection current, current density and voltage. Measuring induced voltages on the surface subject to multiple injection currents, an EIT system produces images of the internal admittivity distribution using an inversion method.

Mathematical theory has been developed to support such an EIT system especially for the unique identification of the conductivity $\sigma$ from knowledge of all possible boundary current-to-voltage data at low frequencies where we can assume $\gamma \approx \sigma$ ?????????. After the early attempt to build an EIT system (Barber and Brown 1984), numerous studies have accumulated knowledge and experience summarized in the recently published book on EIT (Holder 2005). The nonlinear inverse problem in EIT suffers from its ill-posedness related with lack of enough measurable information and insensitivity of measured data to a local change of an internal admittivity value. Though there exist numerous image reconstruction algorithms ?????????????, it is difficult to reconstruct accurate admittivity images with a high spatial resolution in a practical setting where modeling and measurement errors are unavoidable. In this chapter, we focus on robust image reconstructions that may overcome technical difficulties of the ill-posedness.

1.2 Measurement Method and Data

1.2.1 Conductivity and Resistance

We consider a cylinder filled with a saline. The saline contains mobile charges of ions and their migration under an external electric field characterizes its conductivity $\sigma$ in siemens per meter (S/m). Attaching two metal electrodes on the top and bottom surfaces, we measure its resistance $R$ in Ohm (Ω). Neglecting interfacial phenomena between electrodes and the saline, the resistance $R$ is denoted as

$$ R = \frac{1}{\sigma} \frac{L}{A} \tag{1.1} $$

where $L$ and $A$ are the length in meter (m) and cross-sectional area in m$^2$ of the cylinder, respectively. If we inject dc current $I$ in ampere (A), the induced dc voltage $V$ in volt (V) follows Ohm’s law as

$$ V = RI. \tag{1.2} $$

Injecting a known dc current $I$ and measuring the induced dc voltage, we may find the resistance $R$ as is done in an electrical multimeter. If we have geometrical information of $L$ and $A$, we can find the conductivity $\sigma$. For some material like biological tissues, we denote the conductivity as $\sigma_\omega$ to emphasize its frequency dependence. We may measure $\sigma_\omega$ by injecting a sinusoidal current $i(t) = I \cos \omega t$ to measure the induced ac voltage $v(t) = V \cos \omega t$ where $t$ is the time in second (s). Assuming a linear component, the resistance $R$ at $\omega$ also follows Ohm’s law as

$$ v(t) = Ri(t) = RI \cos \omega t = \frac{1}{\sigma_\omega} \frac{L}{A} I \cos \omega t. \tag{1.3} $$

Note that the current and voltage are in phase. Repeating this measurement for multiple frequencies, we may get a conductivity spectrum which plots conductivity $\sigma_\omega$ as a function of frequency $\omega$. 

1.2.2 Permittivity and Capacitance

We consider a dielectric sandwiched by two parallel metal plates. When we apply a dc voltage $V$ between the plates, it induces an electric field inside the dielectric. The dielectric contains immobile charges and their polarization or rotations in the electric field produces surface charges $Q$ and $-Q$ in coulomb (C). The induced charge is proportional to the applied voltage as

$$Q = CV$$  \hspace{0.5cm} (1.4)

where the proportionality constant $C$ is called the capacitance in C/V or farad (F). The capacitance $C$ between two plates is given by

$$C = \epsilon \frac{A}{d}$$  \hspace{0.5cm} (1.5)

where $\epsilon$ is the permittivity in F/m, $A$ the surface area and $d$ the gap between the plates. The permittivity is a material property determined by the polarization of the dielectric under an external electric field. For most dielectrics including biological tissues, the permittivity changes with frequency and we denote it as $\epsilon_\omega$.

If we assume a perfect dielectric, there is no mobile charge and its conductivity $\sigma$ is 0 S/m. Injecting dc current $I$ through the dielectric, we get 0 dc voltage across it. If we apply a sinusoidal voltage $v(t) = V \sin \omega t$, there occurs an ac displacement current through the dielectric due to time-varying polarizations with the frequency $\omega$:

$$i(t) = C \frac{dv(t)}{dt} = \omega CV \cos \omega t = I \cos \omega t.$$  \hspace{0.5cm} (1.6)

Note that the current and voltage are out of phase by $90^\circ$ or the voltage is in the quadrature of the current. Assuming that there is no polarization initially, we can express the induced voltage $v(t)$ subject to an injection current $i(t)$ as

$$v(t) = \frac{1}{C} \int_0^t i(\tau) \, d\tau = \frac{I}{\omega C} \sin \omega t = \frac{I}{\omega \epsilon_\omega A} \cos \left( \omega t - \frac{\pi}{2} \right).$$  \hspace{0.5cm} (1.7)

With known $\omega$ and $I$, we may find the capacitance $C$ in F which equals to A·s/V or s/Ω. If we have geometrical information of $A$ and $d$, we can find the permittivity $\epsilon_\omega$ in F/m. Repeating this measurement for multiple frequencies, we may get a permittivity spectrum which plots permittivity $\epsilon_\omega$ as a function of frequency $\omega$.

1.2.3 Phasor and Impedance

Given an electrically conducting object with both mobile and immobile charges, we may view it as a mixture of resistors and capacitors. In this section, we adopt a circuit model using lumped elements since it provides intuitive understanding about the continuum model. Let’s consider a series $RC$ circuit in Fig. 1.1(a). Injecting a sinusoidal current $i(t) = I \cos \omega t$, we can express the induced voltage $v(t)$ across the series connection of $R$ and $C$ as

$$v(t) = RI(t) + \frac{1}{C} \int_0^t i(\tau) \, d\tau = RI \cos \omega t + \frac{I}{\omega C} \sin \omega t = V \cos (\omega t + \theta)$$  \hspace{0.5cm} (1.8)
\[
V = I \sqrt{R^2 \frac{1}{\omega^2 C^2}} \quad \text{and} \quad \theta = -\arctan \frac{1}{\omega RC}.
\]

Noting that there is no change in frequency between current and voltage for all linear components, we adopt the phasor notation. The current and voltage phasors are defined as complex numbers of \(I = I e^{j\omega t}\) and \(V = V e^{j\omega t}\), respectively. Using the phasor notation, we can handle the relation between time-functions \(i(t)\) and \(v(t)\) as an algebraic equation instead of the integro-differential equation of time.

We define the impedance \(Z\) in \(\Omega\) as the ratio of the voltage phasor \(V\) to the current phasor \(I\) and it is a measure of the total opposition against a circuit to flow through a component or a collection of components. For the case of the series \(RC\) circuit, the total impedance is

\[
Z = Z = Z = \quad \text{where} \quad V = I \sqrt{R^2 \frac{1}{\omega^2 C^2}} \quad \text{and} \quad \theta = -\arctan \frac{1}{\omega RC}.
\]

The real part of \(Z\) is the resistance in \(\Omega\) and its imaginary part is the reactance also in \(\Omega\). Note that for a single resistor \(R\), its impedance \(Z_R = R\). For a single capacitor \(C\), \(Z_C = \frac{1}{j\omega C}\).

**Example 1.2.1** For a series \(RC\) circuit in Fig. 1.1(a) with \(R=10\ k\Omega\) and \(C=0.1\ \mu F\), compute its impedance \(Z\) at the frequency of 10 kHz. For \(i(t) = 10 \cos(2\pi \times 10^4 t)\ mA\), find \(v(t)\) and plot both \(i(t)\) and \(v(t)\) for \(0 \leq t \leq 0.5\ ms\).

**Example 1.2.2** For a parallel \(RC\) circuit in Fig. 1.1(b) with \(R=10\ k\Omega\) and \(C=0.1\ \mu F\), compute its impedance \(Z\) at the frequency of 10 kHz. For \(i(t) = 10 \cos(2\pi \times 10^4 t)\ mA\), find \(v(t)\) and plot both \(i(t)\) and \(v(t)\) for \(0 \leq t \leq 0.5\ ms\).

**Example 1.2.3** For a series-parallel \(RC\) circuit in Fig. 1.1(c) with \(R_1=500\ \Omega\), \(R_2=30\ k\Omega\) and \(C=50\ nF\), compute its impedance \(Z\) at the frequency of 10 kHz. For \(i(t) = 10 \cos(2\pi \times 10^4 t)\ mA\), find \(v(t)\) and plot both \(i(t)\) and \(v(t)\) for \(0 \leq t \leq 0.5\ ms\).

**Example 1.2.4** For a series \(RC\) circuit in Fig. 1.1(a) with \(R=10\ k\Omega\) and \(C=0.1\ \mu F\), plot the magnitude and phase of its impedance \(Z = Z = Z = \theta\) in the frequency range of 1 Hz to 1 MHz.
Example 1.2.5 For a parallel $RC$ circuit in Fig. 1.1(b) with $R=10$ kΩ and $C=0.1\mu F$, plot the magnitude and phase of its impedance $Z = Z\angle \theta$ in the frequency range of 1 Hz to 1 MHz.

Example 1.2.6 For a series-parallel $RC$ circuit in Fig. 1.1(c) with $R_1=500$ Ω, $R_2=30$ kΩ and $C=50$ nF, plot the magnitude and phase of its impedance $Z = Z\angle \theta$ in the frequency range of 1 Hz to 1 MHz.

1.2.4 Admittivity and Trans-impedance

When we consider a material including both mobile and immobile charges, its electrical property is expressed as the admittivity $\gamma$ in S/m. To express its frequency dependence, we denote it as $\gamma_\omega = \sigma_\omega + i\omega\epsilon_\omega$. Note that $\sigma_\omega$ and $\omega\epsilon_\omega$ have the same unit of S/m. We now assume a cylinder filled with a biological tissue whose admittivity is $\gamma_\omega$. The impedance $Z$ between the top and bottom surfaces is

$$Z = \frac{1}{\sigma_\omega + i\omega\epsilon_\omega} \frac{L}{A} = \frac{L}{\sigma_\omega A} \cdot \frac{1 - \frac{i\omega\epsilon_\omega}{\sigma_\omega}}{1 + \left(\frac{\omega\epsilon_\omega}{\sigma_\omega}\right)^2} \tag{1.11}$$

where $L$ and $A$ are the length and cross-sectional area of the cylinder, respectively. If $\frac{\omega\epsilon_\omega}{\sigma_\omega} \ll 1$, that is, $\sigma_\omega \gg \omega\epsilon_\omega$, $Z \approx \frac{L}{\sigma_\omega A} = R$ and the material is resistive. If $\frac{\omega\epsilon_\omega}{\sigma_\omega} \gg 1$, that is, $\sigma_\omega \ll \omega\epsilon_\omega$, $Z \approx -i\frac{L}{\sigma_\omega A} = i\omega C$ and the material is reactive or capacitive.

Most biological tissues are resistive at low frequencies of less than 1 kHz, for example. Since the capacitive term is not negligible beyond 1 kHz, we will denote the admittivity of a biological tissue at a position $r$ as $\gamma_\omega(r)$. We assume an electrically conducting domain $\Omega$ with its admittivity distribution $\gamma_\omega(r)$ as illustrated in Fig. 1.3. Attaching $E$ electrodes $\mathcal{E}^1, \mathcal{E}^2, \cdots, \mathcal{E}^E$, we inject current $i^j(t) = I^j \cos \omega t$ through a pair of electrodes $\mathcal{E}^j$ and $\mathcal{E}^{j+1}$. Between other pair of electrodes $\mathcal{E}^k$ and $\mathcal{E}^{k+1}$, we measure the induced voltage $v^k(t) = V^k \cos(\omega t + \theta^k)$. We define the trans-impedance from the $j$th port to the $k$th port as

$$Z^j,k = \frac{V^k}{I^j} = \frac{V^k}{I^j} \angle \theta^k. \tag{1.12}$$
In section 1.4, we will show that the admittivity distribution \( \gamma(\omega, r) \), domain geometry and electrode configuration affect the trans-impedance \( Z_{j,k} \). The reciprocity principle explained in section 1.4 indicates that \( Z_{j,k} = Z_{k,j} \).

### 1.2.5 Electrode Contact Impedance

In order to inject current and measure voltage, we use electrodes. An electrode is made of a highly conductive material such as copper, silver, platinum and other metals. Carbon is also used to make a flexible electrode though its conductivity is not as large as metallic conductors. When the electrode makes a contact with an electrolyte or the skin of a living subject, the interface can be modeled as a contact impedance and a contact potential in series. The contact impedance includes both resistive and reactive terms and its typical circuit model is the series-parallel \( RC \) circuit in Fig. 1.1(c). As long as the interface is mechanically stable, the contact potential is stable and less than 1 V for most biopotential electrodes.

We consider a method to measure the impedance \( Z \) of the cylinder with its homogeneous admittivity \( \gamma(\omega, r) = \gamma_0 \). Attaching a pair of electrodes at the top and bottom surfaces, we inject current \( I \) with \( \omega \) from the top to the bottom electrodes. Denoting the contact impedances of the top and bottom electrodes as \( Z_{c1} \) and \( Z_{c2} \), respectively, the induced voltage will be expressed as

\[
V = I(Z_{c1} + Z + Z_{c2})
\]  

(1.13)

assuming that no current flows into the ideal voltmeter. We can ignore the dc contact potential since we measure only the induced voltage at the frequency \( \omega \). Using this two-electrode or bipolar method shown in Fig. 1.4(a), it is not possible to extract only \( Z \) since two contact impedances are in series with \( Z \).

By attaching another pair of electrodes around the cylinder near its top and bottom, we inject current through the first pair and measure the induced voltage between the second pair as shown in Fig. 1.4(b). Using a well-designed voltmeter, we may safely assume that there is no current flowing through the second pair of voltage-sensing electrodes. This means that the voltmeter sees only the voltage drop across the impedance of the cylinder \( Z \) between the
Figure 1.4 Impedance measurements using (a) two-electrode or bipolar method and (b) four-electrode or tetrapolar method. No current flows through the ideal voltmeter.

second pair of electrodes as

\[ V =IZ. \]  

This four-electrode or tetrapolar method allows us remove the effects of contact impedances in bioimpedance measurements.

1.2.6 EIT System

We consider an imaging domain \( \Omega \) with its admittivity distribution \( \gamma_\omega (\mathbf{r}) = \sigma_\omega (\mathbf{r}) + i\omega \epsilon_\omega (\mathbf{r}) \). We attach \( E \) electrodes \( \mathcal{E}^1, \mathcal{E}^2, \ldots, \mathcal{E}^E \) on its boundary \( \partial \Omega \). We use an EIT system equipped with current sources and voltmeters to measure trans-impedances or equivalent current-voltage data sets. We may do this for multiple frequencies at different times. A typical EIT system comprises one or multiple current sources, one or multiple voltmeters, optional switching networks, a computer system and a dc power supply. The computer controls current sources, voltmeters and switches to acquire current-voltage data sets. It produces images of \( \sigma_\omega (\mathbf{r}) \) and/or \( \omega \epsilon_\omega (\mathbf{r}) \) by applying an image reconstruction algorithm to the data sets.

There exist several EIT systems with different design concepts and technical details in their implementations. The number of electrodes used in available EIT systems ranges from 8 to 256. The human interface gets complicated with a large number of electrodes and lead wires. With a large number of electrodes, the induced voltage between a pair of electrodes tends to become small since the gap between them gets smaller. In chest imaging, 8 or 16 electrodes are commonly used while more electrodes are used in head or breast imaging.

We may classify recent EIT systems into two types. The first is characterized as one current source with switching networks. In this case, current is sequentially injected between a chosen pair of electrodes and there always exists only one active current source. The second type uses multiple current sources without any switching for current injection. With this type, one may inject a pattern of current through multiple electrodes using multiple active current sources. The sum of currents from all active current sources must be zero. In most EIT systems belonging to both types, voltages between many electrode pairs are simultaneously measured using multiple voltmeters. Typical examples of the first and second types are Mk3.5 from Sheffield \cite{Mk3.5} and ACT3 from RPI \cite{ACT3}, respectively. Boone et al. \cite{Booneetal} and Saulnier \cite{Saulnier} summarized numerous techniques in the development of EIT system. Figure 1.5 shows examples of EIT system and its use for chest imaging \cite{ChestImaging}. 

Figure 1.5 (a) and (b) are KHU Mark1 16- and 32-channel multi-frequency EIT systems, respectively. (c) Chest imaging using a 16-channel KHU Mark1.

The range of the trans-impedance is from a few mΩ to tens of Ω depending on the imaging object, number of electrodes and their configuration. Assuming injection currents of 1 mA\textsubscript{rms}, for example, induced voltages are in the range of a few µV to tens of mV. Allowing a noise level of 1% of the smallest voltage, we should restrict the level below 0.1 µV and this requires the state-of-the-art electronic instrumentation technology. Modern EIT systems usually acquire a complete set of current-voltage data within 10 ms for frequencies higher than 10 kHz. Temporal resolutions could be higher than 20 frames/s using a fast image reconstruction algorithm.

### 1.2.7 Data Collection Protocol and Data Set

A data collection protocol defines a series of injection currents and corresponding voltage measurements. In this section, we introduce only the neighboring protocol. Others will be discussed in section 1.5, where we examine the sensitivity of a voltage measurement to a local change of an internal admittivity value. We assume an EIT system with \( E \) electrodes. Injecting the \( j \)th current between an adjacent pair of electrodes \( \mathcal{E}_j \) and \( \mathcal{E}_{j+1} \), we measure induced boundary voltages between all neighboring pairs of electrodes \( \mathcal{E}_k \) and \( \mathcal{E}_{k+1} \) for \( k = 1, 2, \cdots, E \). Any index number must be understood as a modulus of the maximal value of the index number. We define this data set as a projection, which is the term with its origin from the X-ray CT area. Repeating this for all pairs of current injection electrodes with \( j = 1, 2, \cdots, E \), we can obtain a full set of data from \( E \) projections. The \( k \)th boundary voltage phasor in the \( j \)th injection current or the \( j \)th projection is denoted as

\[
V^{j,k} = V^j|_{\mathcal{E}_k} - V^j|_{\mathcal{E}_{k+1}}
\]

for \( j, k = 1, 2, \cdots, E \). Since the number of injection currents or projections is \( E \) and the number of boundary voltage phasors per projection is also \( E \), the full data set includes \( E^2 \) boundary voltage phasors.

From the reciprocity theorem introduced in section 1.4 and Kirchhoff’s voltage law, only \( E \times (E - 1)/2 \) boundary voltage data are independent. This is the maximal amount of measurable information using \( E \) electrodes regardless of an adopted data collection protocol.
This imposes a fundamental limit in the maximal number of pixel values of a reconstructed admittivity image using any inversion method. For each injection current between a chosen pair of neighboring electrodes, boundary voltage data between three adjacent pairs of electrodes are involved with at least one current-injection electrode. These three voltage data contain the effects of unknown contact impedances between the electrodes and the skin. We may discard or include these data depending on the way contact impedances are treated in a chosen inversion method and electrode model discussed in section 1.4.

Figure 1.6 shows examples of the neighboring protocol assuming a 16-channel EIT system. For each projection, 13 boundary voltage phasors between adjacent pairs of electrodes are measured to adopt the four-electrode method. In this example, the number of projections is 16 and the total number of measured boundary voltage phasors is $16 \times 13 = 208$. Among them, only 104 boundary voltage phasors carry independent information. This indicates that the best spatial resolution of a reconstructed admittivity image is about 10% of the size of the imaging object using a 16-channel EIT system with the neighboring protocol. Using a 32-channel system, we may improve it to 5%.

We can collect boundary voltage data at multiple frequencies for a certain period of time. Assuming that we collected $E^2$ number of boundary voltage data at each sampling time $t$ and frequency $\omega$, we can express the boundary voltage data set in a matrix form as

$$
F_{t,\omega} = \begin{bmatrix}
V_{1,1}^{1,1} & V_{1,2}^{1,1} & \cdots & V_{1,1}^{1,E} \\
V_{2,1}^{1,1} & V_{2,2}^{1,1} & \cdots & V_{2,1}^{1,E} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
V_{E,1}^{1,1} & V_{E,2}^{1,1} & \cdots & V_{E,1}^{1,E} \\
V_{1,1}^{t,\omega} & V_{1,2}^{t,\omega} & \cdots & V_{1,1}^{t,\omega} \\
V_{2,1}^{t,\omega} & V_{2,2}^{t,\omega} & \cdots & V_{2,1}^{t,\omega} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
V_{E,1}^{t,\omega} & V_{E,2}^{t,\omega} & \cdots & V_{E,1}^{t,\omega}
\end{bmatrix}
$$

← 1st projection
← 2nd projection
\ldots
\ldots
\vdots
\vdots
← Eth projection.

(1.15)
Alternatively, we may adopt a column vector representation as

$$\mathbf{F}_{t, \omega} = \begin{bmatrix} V_{t, \omega}^{1, 1} & \cdots & V_{t, \omega}^{1, E} & \cdots & V_{t, \omega}^{2, 1} & \cdots & V_{t, \omega}^{2, E} & \cdots & \cdots & V_{t, \omega}^{E, 1} & \cdots & V_{t, \omega}^{E, E} \end{bmatrix}^T$$

(1.16)

where the superscript $T$ means the transpose. This column vector representation will be used in sections where we study image reconstruction algorithms. For $\omega = \omega_1, \omega_2, \cdots, \omega_F$, we may collect $F$ data vectors or matrices for each sampling time $t = t_1, t_2, \cdots, t_N$ of total $N$ times.

1.2.8 **Linearity between Current and Voltage**

Before we move on to mathematical topics in EIT, we note the linear relationship between injection currents and boundary voltages. We assume that the internal admittivity distribution $\gamma_\omega(r)$, domain geometry and electrode configuration are all fixed. For an injection current $I^j$ or the $j$th projection, we measure $E$ boundary voltage phasors $V_{j, k}^j$ for $k = 1, 2, \cdots, E$ to form the $j$th projection data vector $\mathbf{V}^j$ as

$$\mathbf{V}^j = \begin{bmatrix} V_{j, 1}^j & \cdots & V_{j, E}^j \end{bmatrix}^T.$$  

(1.17)

We now inject current $I$ as

$$I = \sum_{j=1}^E \alpha^j I^j$$

(1.18)

with some real constants $\alpha^j$ for $j = 1, 2, \cdots, E$. The corresponding projection data vector $\mathbf{V}$ is expressed as

$$\mathbf{V} = \sum_{j=1}^E \alpha^j \mathbf{V}^j.$$  

(1.19)

This stems from the linearity between injection currents and induced voltages when we view the imaging object as a mixture of linear resistors and capacitors.

1.3 **Representation of Physical Phenomena**

We assume an imaging object occupying a domain $\Omega$ with its boundary $\partial \Omega$ and an internal admittivity distribution $\gamma(r)$. Using an $E$-channel EIT system, we attach $E$ surface electrodes $E_j$ for $j = 1, \cdots, E$ on $\partial \Omega$ and inject current $i(t) = I \cos \omega t$ through an adjacent pair of electrodes as shown in Fig. 1.6. We assume that the current source and sink are connected to electrodes $E_j$ and $E_{j+1}$, respectively. The injection current produces internal current density and magnetic flux density distributions, which are dictated by Maxwell’s equations in Table 1.1. Table 1.2 summarizes variables used in Maxwell’s equations.

In the frequency range of a few Hz to MHz, we adopt the elliptic PDE studied in section xxx to describe the forward problem in EIT. From Maxwell’s equations, we derive the elliptic PDE and its boundary conditions. After analyzing the PDE in terms of its min-max property, we formulate the EIT forward problem and its model.
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Table 1.1 Maxwell’s equations for time-varying and time-harmonic fields

<table>
<thead>
<tr>
<th>Name</th>
<th>Time-varying Field</th>
<th>Time-harmonic Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’s law</td>
<td>$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$</td>
<td>$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$</td>
</tr>
<tr>
<td>Gauss’s law for magnetism</td>
<td>$\nabla \cdot \mathbf{H} = 0$</td>
<td>$\nabla \cdot \mathbf{H} = 0$</td>
</tr>
<tr>
<td>Faraday’s law of induction</td>
<td>$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$</td>
<td>$\nabla \times \mathbf{E} = -i\omega \mathbf{B}$</td>
</tr>
<tr>
<td>Ampère’s circuital law</td>
<td>$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$</td>
<td>$\nabla \times \mathbf{H} = \mathbf{J} + i\omega \mathbf{D}$</td>
</tr>
</tbody>
</table>

Table 1.2 Variables to describe time-harmonic and time-varying electromagnetic fields

<table>
<thead>
<tr>
<th>Variable</th>
<th>Meaning</th>
<th>Unit</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{E}(r)$ and $\mathbf{E}(r, t) = \Re{\mathbf{E}(r)e^{i\omega t}}$</td>
<td>electric field intensity</td>
<td>V/m</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{H}(r)$ and $\mathbf{H}(r, t) = \Re{\mathbf{H}(r)e^{i\omega t}}$</td>
<td>magnetic field intensity</td>
<td>A/m</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{D}(r)$ and $\mathbf{D}(r, t) = \Re{\mathbf{D}(r)e^{i\omega t}}$</td>
<td>electric flux density</td>
<td>C/m²</td>
<td>$\mathbf{D} = \varepsilon \mathbf{E}$</td>
</tr>
<tr>
<td>$\mathbf{B}(r)$ and $\mathbf{B}(r, t) = \Re{\mathbf{B}(r)e^{i\omega t}}$</td>
<td>magnetic flux density</td>
<td>T</td>
<td>$\mathbf{B} = \mu \mathbf{H}$</td>
</tr>
<tr>
<td>$\mathbf{J}(r)$ and $\mathbf{J}(r, t) = \Re{\mathbf{J}(r)e^{i\omega t}}$</td>
<td>current density</td>
<td>A/m²</td>
<td>$\mathbf{J} = \sigma \mathbf{E}$</td>
</tr>
<tr>
<td>$\epsilon$ and $\epsilon_0 = 8.85 \times 10^{-12}$ (free space)</td>
<td>permittivity</td>
<td>F/m</td>
<td></td>
</tr>
<tr>
<td>$\mu$ and $\mu_0 = 4\pi \times 10^{-7}$ (free space)</td>
<td>permeability</td>
<td>H/m</td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>conductivity</td>
<td>S/m</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>charge density</td>
<td>C/m³</td>
<td></td>
</tr>
</tbody>
</table>

1.3.1 Derivation of Elliptic PDE

In order to simplify mathematical derivations, we assume that the admittivity $\gamma(r) = \sigma(r) + i\omega\varepsilon(r)$ in $\Omega$ is isotropic and $0 < \sigma, \varepsilon < \infty$. For some biological tissues such as muscles and neural tissues, the isotropy assumption is not valid at low frequencies. We assume that the magnetic permeability $\mu$ of the imaging object is $\mu_0$, the magnetic permeability of the free space.

In the frequency range of a few Hz to MHz, we neglect the Faraday induction to get

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B} \approx 0.$$ 

Since $\mathbf{E}$ is approximately irrotational, it follows from Stokes’s theorem that we can define a potential $u$ between any two points $r_1$ and $r_2$ as

$$u(r_2) - u(r_1) = -\int_{C_{r_1 \rightarrow r_2}} \mathbf{E} \cdot dl,$$

where $C_{r_1 \rightarrow r_2}$ is a curve in $\Omega$ joining the starting point $r_1$ to the ending point $r_2$. The complex potential $u$ satisfies

$$-\nabla u(r) = \mathbf{E}(r) \quad \text{in } \Omega.$$ 

From $\nabla \times \mathbf{H} = \mathbf{J} + i\omega\varepsilon \mathbf{E} = (\sigma + i\omega\varepsilon) \mathbf{E}$, we have the following relation:

$$\nabla \times \mathbf{H}(r) = (\sigma(r) + i\omega\varepsilon(r)) \mathbf{E}(r) = -\gamma(r) \nabla u(r).$$

Since $\nabla \cdot (\nabla \times \mathbf{H}) = 0$, the complex potential $u$ satisfies the following elliptic PDE with a complex parameter $\gamma$:

$$-\nabla \cdot (\gamma(r) \nabla u(r)) = 0 \quad \text{in } \Omega.$$ (1.20)
Note that the complex potential \( u \) is equivalent to the voltage phasor introduced in section 1.2. In the rest of this chapter, we denote \( u \) as the voltage phasor or time-harmonic voltage.

### 1.3.2 Elliptic PDE for Four-electrode Method

Using the four-electrode method, we can neglect the contact impedance introduced in section 1.2. Investigating the boundary \( \partial \Omega \) of the imaging object \( \Omega \) with attached electrodes \( E_k \) with \( k = 1, 2, \cdots, E \), we can observe the followings.

**Current-injection electrodes:** Since the total injection current spreads over each current-injection electrode, \( \int_{E_j} \textbf{J}(r, t) \cdot \textbf{n} ds = \int_{E_{j+1}} \textbf{J}(r, t) \cdot \textbf{n} ds = I \cos \omega t \) where \( \textbf{n} \) is the unit outward normal vector and \( ds \) the surface element on \( \partial \Omega \).

**Boundary without any electrode:** Since the air is an insulator, \( \textbf{J}(r, t) \cdot \textbf{n} = 0 \) on \( \partial \Omega \setminus (\cup_{k=1}^E E_k) \).

**Voltage-sensing electrodes:** Since there is no current flowing into a voltmeter, \( \int_{E_k} \textbf{J}(r, t) \cdot \textbf{n} ds = 0 \) for \( k \in \{1, 2, \cdots, E\} \setminus \{j, j + 1\} \).

**All electrodes:** Since \( u \) is approximately constant on each electrode with a very high conductivity, \( \nabla \times \textbf{J} \approx 0 \) on \( E_k \) for \( k = 1, 2, \cdots, E \).

From these observations, we can derive the following boundary conditions for the time-harmonic potential \( u \) in (1.20).

**BC 1:** \( \int_{E_j} (\gamma \nabla u) \cdot \textbf{n} ds = -\int_{E_{j+1}} (\gamma \nabla u) \cdot \textbf{n} ds = I \).

**BC 2:** \( (\gamma \nabla u) \cdot \textbf{n} = 0 \) on \( \partial \Omega \setminus (\cup_{k=1}^E E_k) \).

**BC 3:** \( \int_{E_k} (\gamma \nabla u) \cdot \textbf{n} ds = 0 \) for \( k \in \{1, 2, \cdots, E\} \setminus \{j, j + 1\} \).

**BC 4:** \( \nabla u \times \textbf{n} \approx 0 \) on \( E_k \) for \( k = 1, 2, \cdots, E \).

We define \( g \) as

\[
 g := \gamma \frac{\partial u}{\partial \textbf{n}} \bigg|_{\partial \Omega} \quad (1.21)
\]

and call it the Neumann data of \( u \). In practice, it is difficult to specify the Neumann data \( g \) in a point-wise sense because only the total injection current \( I \) is known. Note that the Neumann boundary data \( g \) has a singularity along the edge of each electrode and \( g \notin L^2(\partial \Omega) \).

Fortunately, we can prove that \( g \in H^{-1/2}(\partial \Omega) \) by the standard regularity theory in PDE. The total injection current through the electrode \( E_j \) is \( I = \int_{E_j} g ds \). The condition \( \nabla u \times \textbf{n} \approx 0 \) on each electrode ensures that \( u|_{E_k} \) is approximately a constant for each electrode since \( \nabla u \) is normal to its level surface.

Expressing the boundary conditions by \( g \), the time-harmonic voltages \( u \) is governed by

\[
 \begin{cases}
 \nabla \cdot (\gamma(r) \nabla u(r)) = 0 & \text{in } \Omega \\
 \gamma \nabla u \cdot \textbf{n} = g & \text{on } \partial \Omega.
\end{cases}
\quad (1.22)
\]

Since \( g \) is the magnitude of the current density on \( \partial \Omega \) due to the injection current, \( g = 0 \) on \( \partial \Omega \setminus (E_j \cup E_{j+1}) \) and \( \int_{E_j} g ds = I - \int_{E_{j+1}} g ds \). Setting a reference voltage \( u(r_0) = 0 \) for a fixed point \( r_0 \in \Omega \), we can obtain a unique solution \( u \) of (1.22) from \( \gamma \) and \( g \). Note that \( u \) depends on \( \gamma, g \) and the geometry of \( \Omega \). When \( \gamma \) changes with \( \omega \), so does \( u \).
Figure 1.7 (a) An example of an electrically conducting domain with a given conductivity distribution. Numbers inside ellipsoids are conductivity values in S/m. (b) Voltage and current density distribution induced by the injection current. Black and white lines are equipotential and current density streamlines, respectively.

Example 1.3.1 Assume that copper electrodes are perfect conductors and $\omega \epsilon / \sigma = 0$. The potential $u \in \mathbb{R}$ satisfies

\[
\begin{aligned}
\nabla \cdot (\sigma \nabla u) &= 0 \quad \text{in } \Omega \\
I &= \int_{E_j} \sigma \frac{\partial u}{\partial n} \, ds = - \int_{E_{j+1}} \sigma \frac{\partial u}{\partial n} \, ds \\
\int_{E_k} n \cdot (\gamma \nabla u) \, ds &= 0 \quad \text{for } k \in \{1, 2, \cdots, E\} \setminus \{j, j+1\} \\
\nabla u \times n &= 0 \quad \text{on } \bigcup_{k=1}^{E_k} E_k \\
\sigma \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus \bigcup_{k=1}^{E_k} E_k.
\end{aligned}
\] (1.23)

The above nonstandard boundary value problem is well-posed and has a unique solution within $H^1(\Omega)$ up to a constant. Figure 1.7 illustrates a numerical example.

Example 1.3.2 Assume that $u$ is a solution of (1.23). Then $\nabla u$ is singular at the edge of a current-injection electrode. To estimate this singularity, we consider a simplified model $\Omega = \mathbb{R}^3_z := \{ x : z < 0 \}$ and $E = \{(x, y, 0) : \sqrt{x^2 + y^2} < 1\}$. Let $w$ be the $H^1(\Omega)$-solution of the following mixed boundary value problem:

\[
\begin{aligned}
-\nabla^2 w &= 0 \quad \text{in } \Omega \\
\frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus E \\
w|_{E} &= 1.
\end{aligned}
\] (1.24)

Let $g = \nabla w \cdot n|_{\partial \Omega}$. Prove that $g$ satisfies the integral equation:

\[
1 = \frac{1}{2\pi} \int_{\Gamma} \frac{g(x', y')}{\sqrt{(x-x')^2 + (y-y')^2}} \, dx' \, dy' \quad \text{if } \sqrt{x^2 + y^2} < 1.
\] (1.25)

Find a representation formula for $w$ and find the behavior of $g$ near the circular edge of the electrode $E$. 


Solution. Assume that \( g \) satisfies (1.25). Define
\[
U(r) = \int_{\gamma} \mathcal{N}(r, r') g(r') dr' \quad r \in \Omega
\]
where \( \mathcal{N} \) is the Neumann function
\[
\mathcal{N}(r, r') := \frac{1}{4\pi} \left( \frac{1}{(x-x')^2 + (y-y')^2 + (z-z')^2} + \frac{1}{(x-x')^2 + (y-y')^2 + (z+z')^2} \right).
\]
It is easy to see that \( U \in H^1(\Omega) \) satisfies
\[
\nabla^2 U = 0 \quad \text{in} \quad \Omega, \quad U|_{\partial \Omega} = 1 \quad \text{and} \quad \frac{\partial U}{\partial n}|_{\partial \Omega \setminus \mathcal{E}} = 0.
\]
It follows from the uniqueness theory that the solution \( w \) of (1.24) must be \( U = w \) in \( \Omega \). Since \( g \) is radial, \( g(r) = g(\sqrt{x^2 + y^2}) \) satisfies
\[
1 = \int_0^1 \phi(t, r) g(r) dr \quad (0 \leq t < 1)
\]
where \( \phi(t, r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r}{\sqrt{r^2 - 2rt \cos \theta + t^2}} d\theta \). Noting that \( 0 = \frac{d}{dt} \int_0^1 \phi(t, r) g(r) dr \) for \( 0 < t < 1 \), one can show that \( \lim_{t \to 1^{-}} |g(r)| = \infty \).

1.3.3 Elliptic PDE for Two-electrode Method

When we adopt the two-electrode method where we measure voltages on current-injection electrodes, we must take into account of the contact impedance. We introduce the complete electrode model (Cheng et al. 1989, Somersalo et al. 1992, Vauhkonen et al. 1999) where the complex potential \( u \) satisfies
\[
\begin{align*}
\nabla \cdot (\gamma \nabla u) &= 0 \quad \text{in} \quad \Omega \\
(u + z_k \gamma \frac{\partial u}{\partial n})|_{\partial \Omega \setminus \cup_{k=1}^{E} \mathcal{E}_k} &= U_k, \quad k = 1, \ldots, E \\
\gamma \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial \Omega \cup \cup_{k=1}^{E} \mathcal{E}_k \\
\int_{\mathcal{E}_j} \gamma \frac{\partial u}{\partial n} ds &= I = -\int_{\cup_{k=1}^{E}} \gamma \frac{\partial u}{\partial n} ds
\end{align*}
\]
(1.26)
where \( z_k \) is the contact impedance of the \( k \)th electrode \( \mathcal{E}_k \) and \( U_k \) is the voltage on \( \mathcal{E}_k \). Setting a reference voltage having \( \sum_{k=1}^{E} U_k = 0 \), we can obtain a unique solution \( u \) of (1.26).

In this case, measured boundary voltages are
\[
V^{j,1} := U_j^1 - U_{E}^1, \quad V^{j,2} := U_j^2 - U_{E}^2, \quad \ldots, \quad V^{j,E} := U_j^E - U_{E-1}^E.
\]

Using an \( E \)-channel EIT system, we may inject \( E \) number of currents through adjacent pairs of electrodes and measure the following voltage data set:
\[
F = [V^{1,1}, \ldots, V^{1,E}, \ldots, V^{E,1}, \ldots, V^{E,E}]^T \in \mathbb{C}^{2E^2}.
\]
The voltage data are influenced by contact impedances whose values are unknown. Since the reciprocity principle of \( V^{k,j} = V^{j,k} \) in section 1.4 still holds, \( F \) contains at most \( E(E - 1)/2 \) number of independent data.
1.3.4 Min-max Property of Complex Potential

The variational form of the problem (1.22) with the Neumann boundary condition is
\[ \int_{\Omega} \gamma \nabla u \cdot \nabla \phi \, dr = \int_{\partial \Omega} g \phi \, ds \quad \text{for all } \phi \in H^1(\Omega). \] (1.27)

According to the Lax-Milgram theorem in section xxx, for a given \( g \in H^{-1/2}(\partial \Omega) \) with \( \int_{\partial \Omega} g \, ds = 0 \), there exists a unique solution \( u \in H^1(\Omega) \) with \( \int_{\partial \Omega} g u \, ds = 0 \) satisfying (1.27). When \( \omega = 0 \), we can figure out the global structure of \( u \in \mathbb{R} \) using its weighted mean value property, maximum principle and the minimization property of the corresponding energy functional:
\[ \int_{\partial \Omega} g \, u \, ds = \min_{w \in A_g} \int_{\Omega} |\nabla w|^2 \, dr, \quad A_g = \left\{ w \in H^1(\Omega) : \frac{\partial w}{\partial n} |_{\partial \Omega} = g \right\}. \] (1.28)

When \( \omega > 0 \), the potential \( u \in \mathbb{C} \) does not have the minimization property (1.28), mean value property and maximum principle. Denoting \( v = \Re u \) and \( h = \Im u \), \( u = v + i h \) satisfies the following coupled system:
\[
\begin{aligned}
\nabla \cdot \left( \sigma \nabla v \right) - \nabla \cdot \left( \omega \epsilon \nabla h \right) &= 0 \quad \text{in } \Omega \\
\nabla \cdot \left( \omega \epsilon \nabla v \right) + \nabla \cdot \left( \sigma \nabla h \right) &= 0 \quad \text{in } \Omega \\
\mathbf{n} \cdot (\sigma \nabla v(x) + \omega \epsilon \nabla h(x)) &= g \quad \text{on } \partial \Omega \\
\mathbf{n} \cdot (\sigma \nabla h(x) - \omega \epsilon \nabla v(x)) &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] (1.29)

The complex potential \( u \) has the min-max property (Cherkaev 1995) in the sense that
\[
\int_{\partial \Omega} g \Re\{u\} \, ds = \min_{v \in H^2(\Omega)} \max_{h \in H^2(\Omega)} \int_{\Omega} \left[ \Re\{\gamma\} \left( |\nabla v|^2 - |\nabla h|^2 \right) - 2 \Im\{\gamma\} \nabla v \cdot \nabla h \right] \, dr
\] (1.30)

and
\[
\int_{\partial \Omega} g \Im\{u\} \, ds = \min_{v \in H^2(\Omega)} \max_{h \in H^2(\Omega)} \int_{\Omega} \left[ \Im\{\gamma\} \left( |\nabla v|^2 - |\nabla h|^2 \right) + 2 \Re\{\gamma\} \nabla v \cdot \nabla h \right] \, dr.
\] (1.31)

1.4 Forward Problem and Model

We describe the forward problem of EIT using the Neumann-to-Dirichlet (NtD) data, which depends on the admittivity \( \gamma \). After introducing the continuous NtD data and its theoretical issues, we formulate the discrete NtD data of an \( E \)-channel EIT system.

1.4.1 Continuous Neumann-to-Dirichlet Data

We define the continuous NtD data set \( \Lambda_\gamma \) as
\[
\Lambda_\gamma : \ H_o^{-1/2}(\partial \Omega) \to H_o^{1/2}(\partial \Omega) \quad g \mapsto u^0_\gamma |_{\partial \Omega}
\] (1.32)
where \( u_g^\gamma \) is the unique solution of Neumann boundary value problem of

\[
\begin{cases}
\nabla \cdot (\gamma(r) \nabla u_g^\gamma(r)) = 0 & \text{in } \Omega \\
-\gamma \nabla u_g^\gamma \cdot n|_{\partial \Omega} = g, \quad \int_{\partial \Omega} u \, ds = 0.
\end{cases}
\]

This NtD data \( \Lambda_\gamma \) includes all possible Cauchy data. With this full data set, the forward problem of EIT is modeled as the map

\[
\gamma \rightarrow \Lambda_\gamma
\]

and the inverse problem is to invert the map in (1.35).

There are two major theoretical questions regarding the map.

**Uniqueness:** Is the map \( \gamma \rightarrow \Lambda_\gamma \) injective?

**Stability:** Find the estimate of the form:

\[
\| \log \gamma_1 - \log \gamma_2 \| \ast \leq \Psi \left( \| \Lambda_{\gamma_1} - \Lambda_{\gamma_2} \|_L \right)
\]

where \( \| \cdot \|_\ast \) is an appropriate norm for the admittivity, \( \Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a continuously increasing function with \( \Psi(0) = 0 \) and \( \| \cdot \|_L \) is the operator norm on \( L(H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)) \).

The NtD data \( \Lambda_\gamma \) is closely related with the Neumann function restricted on \( \partial \Omega \). The Neumann function \( N_\gamma(r, r') \) is the solution of the following Neumann problem: for each \( r, \)

\[
\begin{cases}
\nabla \cdot (\gamma(r') \nabla N_\gamma(r, r')) = \delta(r' - \cdot) & \text{in } \Omega \\
\gamma \nabla N_\gamma(r, \cdot) \cdot n = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( \delta \) is the Dirac delta function. With the use of the Neumann function \( N_\gamma(r, r') \), we can represent \( u_g^\gamma(r) \) in terms of the singular integral:

\[
\begin{align*}
u_g^\gamma(r) & = \int_{\Omega} \delta(r - r') \, u_g^\gamma(r') \, dr' \\
& = \int_{\Omega} \nabla \cdot (\gamma(r') \nabla N_\gamma(r, r')) \, u_g^\gamma(r') \, dr' \\
& = -\int_{\Omega} \gamma(r') \nabla N_\gamma(r, r') \cdot \nabla u_g^\gamma(r') \, dr' \\
& = \int_{\partial \Omega} N_\gamma(r, r') \, g(r') \, ds_{r'}.
\end{align*}
\]

Since \( \Lambda_\gamma \) is the restriction of \( u_g^\gamma \) to the boundary \( \partial \Omega \), it can be represented as

\[
\Lambda_\gamma[g](r) = \int_{\partial \Omega} N_\gamma(r, r') \, g(r') \, ds_{r'}, \quad r \in \partial \Omega.
\]

The kernel \( N_\gamma(r, r') \) with \( r, r' \in \partial \Omega \) can be viewed as an expression of the NtD data \( \Lambda_\gamma \).

Note that the \( \Lambda_\gamma \) is sensitive to a change in the geometry of the surface \( \partial \Omega \) since \( N_\gamma(r, r') \) is singular at \( r = r' \).

For the uniqueness in a three-dimensional problem, Kohn and Vogelius (1985) showed the injectivity of \( \gamma \rightarrow \Lambda_\gamma \) if \( \gamma \) is piecewise analytic. Sylveste and Uhlmann (1987) showed the injectivity if \( \gamma \in C^\infty(\Omega) \). The smoothness condition on \( \gamma \) and \( \partial \Omega \) has been relaxed by several researchers (Nachman 1988, Nachmann et al 1988, Alessandrini 1990, Isakov 1991).
For a two-dimensional problem, Nachman (1996) proved the uniqueness under some smoothness conditions on $\gamma$ and provided a constructive way of recovering $\gamma$. Based on Nachman’s proof on the two-dimensional global uniqueness, Siltanen et al (2000) developed the D-bar algorithm which solves the full nonlinear EIT problem without iteration. Brown and Uhlmann (1997) proved the uniqueness for $\gamma \in W^{1,p}, p > 1$. Astala and Päivärinta (2006) proved the uniqueness for $\gamma \in L^\infty$.

To reconstruct $\gamma$ by inverting the map (1.35), it would be ideal if the full continuous Neumann-to-Dirichlet data $\Lambda_\gamma$ are available. In practice, it is not possible to get them due to a limited number of electrodes with a finite size. It is also difficult to capture the correct geometry of $\partial \Omega$ at a reasonable cost. The map in (1.35) is highly nonlinear and insensitive to a local change of $\gamma$ as explained in section 1.4.3. All of these hinder stable reconstructs of $\gamma$ with a high spatial resolution.

1.4.2 Discrete Neumann-to-Dirichlet Data

We assume an EIT system using $E$ electrodes $E_j$ for $j = 1, 2, \cdots, E$. The isotropic admittivity distribution in $\Omega$ is denoted as $\gamma(r)$. The complex potential $u$ in (1.22) subject to the $j$th injection current between $E_j$ and $E_{j+1}$ is denoted as $u^j$ and it approximately satisfies the following Neumann boundary value problem:

\[
\begin{aligned}
&\nabla \cdot (\gamma(r) \nabla u^j(r)) = 0 \quad \text{in } \Omega \\
&-\gamma \nabla u^j \cdot n = g^j \quad \text{on } \partial \Omega
\end{aligned}
\]  

(1.37)

where $\int_{E_j} g^j ds = I = -\int_{E_{j+1}} g^j ds$ and the Neumann data $g^j$ is zero on the boundary regions not contacting with the current injection electrodes. Setting a reference voltage at $r_0 \in \Omega$ as $u^j(r_0) = 0$, we can obtain a unique solution $u^j$ from $\gamma$ and $g^j$.

We assume the neighboring data collection protocol in section 1.2 to measure boundary voltages between adjacent pairs of electrodes, $E_k$ and $E_{k+1}$ for $k = 1, 2, \cdots, E$. The $k$th boundary voltage difference subject to the $j$th injection current is denoted as

\[
V^{j,k} = \frac{1}{|E_k|} \int_{E_k} u^j dr - \frac{1}{|E_{k+1}|} \int_{E_{k+1}} u^j dr \quad \text{for } j, k = 1, 2, \cdots, E
\]

(1.38)

where $\frac{1}{|E_k|} \int_{E_k} u^j dr$ can be understood as the average of $u^j$ over $E_k$.

**Lemma 1.4.1** The $k$th boundary voltage difference subject to the $j$th injection current satisfies

\[
\int_\Omega \gamma \nabla u^j : \nabla u^k dx = V^{j,k}[\gamma]
\]

(1.39)

**Proof.** Integration by parts yields

\[
\int_\Omega \delta \gamma \nabla u^j : \nabla u^k dx = \int_{\partial \Omega} u^j g^k ds = V^{j,k}[\gamma]
\]

where the last identity comes from the boundary condition of (1.37). $\square$

Since $V^{j,k}[\gamma]$ is uniquely determined by the distribution of $\gamma$, it can be viewed as a function of $\gamma$. With $E$ projections and $E$ complex boundary voltage data for each projection, we are
Electrical Impedance Tomography provided with $E^2$ complex boundary voltage data, which are expressed in a matrix form as $F$:

$$F[\gamma] := \begin{bmatrix} V^{1,1} & V^{1,2} & \cdots & V^{1,E} \\ V^{2,1} & V^{2,2} & \cdots & V^{2,E} \\ \vdots & \vdots & \ddots & \vdots \\ V^{E,1} & V^{E,2} & \cdots & V^{E,E} \end{bmatrix}$$

$\leftarrow$ 1st projection

$\leftarrow$ 2nd projection

$\leftarrow$ $E$th projection

(1.40)

where $V^{j,k} = V^{j,k}[\gamma]$ for a given $\gamma$.

**Theorem 1.4.2 (Reciprocity of NtD data)** For a given $\gamma$, $V^{j,k}$ in (1.38) satisfies the reciprocity property:

$$V^{j,k} = V^{k,j} \quad \text{for all } k,j = 1,2,\cdots,E.$$  

(1.41)

**Proof.** The reciprocity follows from the identity:

$$V^{j,k} = \int_{\partial\Omega} \gamma \frac{\partial u_j}{\partial n} u_k \, ds = \int_{\Omega} \gamma \nabla u_j \cdot \nabla u_k \, dr = \int_{\partial\Omega} \gamma \frac{\partial u_k}{\partial n} u_j \, ds = V^{k,j}.$$  

**Observation 1.4.1** Assume that $\gamma$ is constant or homogeneous in $\Omega$. Then,

$$\gamma = \frac{\int_{\Omega} \nabla w^j \cdot \nabla w^k \, dr}{V^{j,k}}$$  

(1.42)

where $w^j$ is the solution of (1.37) with $\gamma = 1$.

**Proof.** Since $w^j = \gamma w^j$, we have

$$\gamma = \frac{\int_{\Omega} (\gamma \nabla u^j) \cdot (\gamma \nabla u^k) \, dr}{\int_{\Omega} \gamma \nabla u^j \cdot \nabla u^k \, dr} = \frac{\int_{\Omega} \nabla w^j \cdot \nabla w^k \, dr}{\int_{\Omega} g^j u^k \, dS} = \frac{\int_{\Omega} \nabla w^j \cdot \nabla w^k \, dr}{V^{j,k}}.$$  

The data matrix $F[\gamma]$ in (1.40) can be viewed as a discrete version of the NtD data since it provides all the measurable current-to-voltage relations using the $E$-channel EIT system. With this discrete NtD data set, the forward problem of the $E$-channel EIT is modeled as the map

$$\gamma \rightarrow F[\gamma]$$  

(1.43)

and the inverse problem is to invert the map in (1.43).

The smoothness condition on $\gamma$ should not be a major issue in a practical EIT image reconstruction. For any discontinuous admittivity $\gamma$ and an $E$- channel EIT system, we always find $\tilde{\gamma} \in C^\infty(\bar{\Omega})$ which approximate $\gamma$ in such a way that

$$\sum_{j=1}^{E} \sum_{k=1}^{E} \left| V^{j,k}[\gamma] - V^{j,k}[\tilde{\gamma}] \right| < \text{arbitrary small positive quantity}.$$  

Taking account of inevitable measurement noise in the discrete NtD data and the ill-posedness of its inversion process, we conclude that $\gamma$ and $\tilde{\gamma}$ are not distinguishable in practice.
1.4.3 Nonlinearity between Admittivity and Voltage

As defined in (1.43), the forward model is a map from the admittance to a set of boundary voltage data. From (1.37), we can see that any change in the admittance influences all voltage values. Unlike the linear relation between currents and voltages, the map in (1.43) is nonlinear. Understanding the map should precede designing a method to invert it.

A voltage value at a point inside the domain can be expressed as a weighted average of its neighboring voltages where the weights are determined by the admittance distribution. In this weighted averaging way, information on the admittance distribution is conveyed to the boundary voltage as shown in Fig. 1.7(b). The boundary voltage is entangled with the global structure of the admittance distribution in a highly nonlinear way and we investigate the relation in this section.

We assume that the domain \( \Omega \) is a square in \( \mathbb{R}^2 \) with its conductivity distribution \( \sigma \), that is \( \gamma = \sigma \). We divide \( \Omega \) uniformly into an \( N \times N \) square mesh. Each square element is denoted as \( \Omega_{i,j} \) with its center at \( (x_i, y_j) \) for \( i, j = 1, 2, \cdots, N \). We assume that the conductivity \( \sigma \) is constant in each element \( \Omega_{i,j} \), say \( \sigma_{i,j} \). Let

\[
\Sigma = \{ \sigma \mid \sigma|_{\Omega_{i,j}} = \sigma_{i,j} \text{ constant } \forall i, j = 1, 2, \cdots, N \}.
\]

For a given \( \sigma \in \Sigma \), we can express \( \sigma \) as

\[
\sigma = [\sigma_1, \sigma_2, \cdots, \sigma_{N^2}]^T.
\]

The solution \( u \) of the elliptic PDE in (1.37) with \( \sigma \) in place of \( \gamma \) can be approximated by a vector \( u = [u_1, u_2, \cdots, u_{N^2}]^T \) such that each voltage \( u_k \) for \( k = i + jN \) is determined by the weighted average of four neighboring voltages. To be precise, the conductivity equation

\[
\nabla \cdot (\sigma(r)\nabla u(r)) = 0
\]

can be written as the following discretized form of

\[
u_k = \frac{1}{a_{k,k}} \left( a_{k,k_T} u_{k_T} + a_{k,k_D} u_{k_D} + a_{k,k_R} u_{k_R} + a_{k,k_L} u_{k_L} \right)
\]
with

\[
a_{k,k} = - \sum_d a_{k,k_d} \quad \text{and} \quad a_{k,k_d} = \frac{\sigma_k \sigma_{k_d}}{\sigma_k + \sigma_{k_d}} \quad \text{for } d = T, D, R, L
\]

where \( k_T, k_D, k_R \) and \( k_L \) denote top, down, right and left neighboring points of the \( k \)th point, respectively.

The discretized conductivity equation (1.44) with the Neumann boundary condition can be rewritten as a linear system of equations:

\[
A(\sigma) u = g
\]

where \( g \) is the injection current vector associated with the Neumann boundary data \( g \). Any change in \( \sigma_k \) for \( k = 1, 2, \cdots, N^2 \) spreads its influence to all \( u_k \) for \( k = 1, 2, \cdots, N^2 \) through the matrix \( A(\sigma) \). We should note following implications of the entanglement among \( \sigma_k \) and \( u_k \).
Nonlinearity and insensitivity grow exponentially as the matrix size increases.

**Geometry:** The recursive averaging process in (1.44) with (1.45) makes the influence of a change in \( \sigma_k \) upon \( u_l \) smaller and smaller as the distance between positions of \( \sigma_k \) and \( u_l \) is more increased.

**Nonlinearity:** The recursive averaging process in (1.44) with (1.45) causes a nonlinearity between \( \sigma_k \) and \( u_l \) for all \( k, l = 1, 2, \ldots, N^2 \).

**Interdependence:** The recursive averaging process in (1.44) with (1.45) makes the influence of a change in \( \sigma_k \) upon \( u_l \) affected by all other \( \sigma_m \) with \( m \in \{1, 2, \ldots, N^2\} \setminus \{k\} \).

### 1.5 Sensitivity and Sensitivity Matrix

Before we study inversion methods to invert the map in (1.43), we investigate the sensitivity of a boundary voltage \( V_{j,k}[\gamma] \) to a change in \( \gamma \). We assume that the discrete NtD data \( F \) in (1.40) is available. Since \( F \) can be viewed as a function of \( \gamma \), we denote it by \( F(\gamma) \). In order to explain the sensitivity matrix, we use the vector form \( F(\gamma) \) as

\[
F(\gamma) = [V^{1,1}[\gamma] \ldots V^{1,E}[\gamma] V^{2,1}[\gamma] \ldots V^{2,E}[\gamma] \ldots \ldots V^{E,1}[\gamma] \ldots V^{E,E}[\gamma]]^T
\]  

(1.46)

or

\[
lth \text{ component of } F(\gamma) = f^l(\gamma) = V^{j,k}[\gamma] \text{ for } l = (j - 1) \times E + k
\]

(1.47)

for \( j, k = 1, 2, \ldots, E \).

We assume a reference admittance \( \gamma^0 = \sigma^0 + i\omega\epsilon^0 \) which is a homogeneous admittance minimizing the least square

\[
\|F(\gamma) - F(\gamma^0)\| = \sqrt{\sum_{l=1}^{E^2} |f^l(\gamma) - f^l(\gamma^0)|^2}.
\]

(1.48)

We may assume that \( F(\gamma) \) is a measured data set and \( F(\gamma^0) \) is a computed data set by numerically solving (1.37) with a known \( \gamma^0 \) in place of \( \gamma \).
1.5.1 Perturbation and Sensitivity

We consider $\gamma$ which is different from a known admittivity $\gamma^0$. We inject same currents into the imaging domain with $\gamma$ and $\gamma^0$.

**Lemma 1.5.1** The perturbation $\delta \gamma := \gamma - \gamma^0$ satisfies

$$
\int_\Omega \delta \gamma \nabla u_j \cdot \nabla u_k^{0} \, dr = f^l(\gamma) - f^l(\gamma^0) \quad \text{for} \quad l = (j - 1) \times E + k \tag{1.49}
$$

where $u_k^0$ is the solution of (1.37) with $\gamma^0$ in place of $\gamma$.

**Proof.** Using integration by parts and the reciprocity theorem, we have

$$
\int_\Omega \delta \gamma \nabla u_j \cdot \nabla u_k^{0} \, dr = \int_\Omega \gamma \nabla u_j \cdot \nabla u_k^{0} \, dr - \int_\Omega \gamma^0 \nabla u_j \cdot \nabla u_k^{0} \, dr
$$

$$
= \int_{\partial \Omega} g^j u_k^{0} \, ds - \int_{\partial \Omega} u_j g^k \, ds
$$

$$
= V^{j,k}_{\gamma} - V^{j,k}_{\gamma^0} = V^{j,k}_{\gamma} - V^{j,k}_{\gamma^0}
$$

$$
= f^l(\gamma) - f^l(\gamma^0).
$$

The sensitivity expression in (1.49) provides information about how much boundary voltage changes by the admittivity perturbation $\delta \gamma$.

1.5.2 Sensitivity Matrix

The effects of a perturbation $\delta \gamma$ depend on the position $r$ of the perturbation. In order to construct an explicit expression, we divide the domain $\Omega$ into small subregions and assume that $\gamma, \gamma^0$ and $\delta \gamma$ are constant in each subregion. With this kind of discretization, we can transform (1.49) into a matrix form.

**Observation 1.5.1** We discretize the domain $\Omega$ into $N$ subregions as $\Omega = \bigcup_{n=1}^N T_n$. We assume that $\gamma, \gamma^0$ and $\delta \gamma$ are constants in each $T_n$. We can express (1.49) as

$$
S_{\gamma, \gamma^0} \delta \gamma = F(\gamma) - F(\gamma^0) \tag{1.50}
$$

where

$$
\delta \gamma = \begin{bmatrix}
\delta \gamma_1 \\
\vdots \\
\delta \gamma_N
\end{bmatrix} \in \mathbb{C}^N.
$$
and \( \delta \gamma_n = \delta \gamma |_{T_n} \) is the quantity of \( \delta \gamma \) in \( T_n \). The \( E^2 \times N \) sensitivity matrix \( S_{\gamma, \gamma^0} \) is given by

\[
S_{\gamma, \gamma^0} = \begin{bmatrix}
\vdots \\
((E - 1) \times j + 1)\text{th row} \\
\vdots \\
((E - 1) \times j + E)\text{th row} \\
\vdots \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\vdots \\
\int_{T_1} \nabla u^j \cdot \nabla u^0_k \\
\vdots \\
\int_{T_N} \nabla u^j \cdot \nabla u^0_k \\
\vdots \\
\end{bmatrix}.
\]

Note that the sensitivity matrix depends nonlinearly on the admittivity distributions \( \gamma \) and \( \gamma^0 \).

### 1.5.3 Linearization

We let \( \gamma^0 \) be a variable and make a link between changes in boundary voltages and a small admittivity perturbation \( \delta \gamma \) around \( \gamma^0 \).

**Observation 1.5.2** Assuming the same discretization of the domain \( \Omega \) as explained in the previous section, the admittivity is an \( N \)-dimensional variable. When the perturbation \( \delta \gamma \) is small,

\[
\delta F = F(\gamma^0 + \delta \gamma) - F(\gamma^0) \approx \nabla_{\gamma} F(\gamma^0) \delta \gamma = S_{\gamma^0, \gamma^0} \delta \gamma \tag{1.51}
\]

where \( \nabla_{\gamma} F(\gamma^0) \) can be viewed as a Fréchet derivative of \( F \) with respect to \( \gamma \) at \( \gamma = \gamma^0 \).

**Proof.** Let \( p = (j - 1) \times E + k \). From (1.49),

\[
f^p(\gamma^0 + \delta \gamma) - f^p(\gamma^0) = \int_{\Omega} \delta \gamma \nabla u^j \cdot \nabla u^0_k \, dr
\]

\[
= \int_{\Omega} \delta \gamma \nabla u^j_k \cdot \nabla u^0_k \, dr - \int_{\Omega} \delta \gamma (u^j - u^j_0) \cdot \nabla u^0_k \, dr
\]

\[
\approx \int_{\Omega} \delta \gamma \nabla u^j_k \cdot \nabla u^0_k \, dr + O\left( \|\delta \gamma\|_{L^\infty(\Omega)}^2 \right)
\]

since \( \int_{\Omega} |(u^j - u^j_0)|^2 \, dr = O(\|\delta \gamma\|_{L^\infty(\Omega)}^2) \). Hence,

\[
e_n \cdot \nabla \gamma f^p(\gamma^0) = \lim_{h \to 0} \frac{f^p(\gamma^0 + h \chi_{T_n}) - f^p(\gamma^0)}{h} = \int_{T_n} \nabla u^0_k \cdot \nabla u^k_0 \, dr
\]
where $\chi_{T_n}$ is the characteristic function of $T_n$. The proof follows from the fact that

$$\int_{T_n} \nabla u_0^j \cdot \nabla u_0^k \, dr = (n, p)\text{th component of } S_{\gamma_0, \gamma_0}.$$  

**Observation 1.5.3** We let $S_{\gamma_0} = S_{\gamma_0, \gamma_0}$. The linearized EIT problem is expressed by

$$S_{\gamma_0} \delta \gamma = \delta F$$  

or

$$\begin{bmatrix}
  \int_{T_1} \nabla u_0^1 \cdot \nabla u_0^1 \\
  \int_{T_1} \nabla u_0^1 \cdot \nabla u_0^2 \\
  \vdots \\
  \int_{T_1} \nabla u_0^1 \cdot \nabla u_0^N \\
  \int_{T_N} \nabla u_0^1 \cdot \nabla u_0^1 \\
  \int_{T_N} \nabla u_0^1 \cdot \nabla u_0^2 \\
  \vdots \\
  \int_{T_N} \nabla u_0^1 \cdot \nabla u_0^N \\
  \vdots \\
  \vdots \\
  \vdots \\
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \vdots \\
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \vdots \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_{E} \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_{E} \\
  \vdots \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_{E} \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_{E} \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_{E} \\
  \vdots \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_{E}
\end{bmatrix} \begin{bmatrix}
  \delta \gamma_1 \\
  \vdots \\
  \delta \gamma_N
\end{bmatrix} = \begin{bmatrix}
  \int_{T_1} \nabla u_0^1 \cdot \nabla u_0^1 \\
  \int_{T_1} \nabla u_0^1 \cdot \nabla u_0^2 \\
  \vdots \\
  \int_{T_1} \nabla u_0^1 \cdot \nabla u_0^N \\
  \int_{T_N} \nabla u_0^1 \cdot \nabla u_0^1 \\
  \int_{T_N} \nabla u_0^1 \cdot \nabla u_0^2 \\
  \vdots \\
  \int_{T_N} \nabla u_0^1 \cdot \nabla u_0^N \\
  \vdots \\
  \vdots \\
  \vdots \\
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \vdots \\
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \vdots \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_{E-1} \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_{E} \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_{E} \\
  \vdots \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_{E} \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_{E} \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_{E} \\
  \vdots \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_{E}
\end{bmatrix} \begin{bmatrix}
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_0^{E-1} \\
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_0^{E-1} \\
  \vdots \\
  \int_{T_1} \nabla u_{E-1} \cdot \nabla u_0^{E-1} \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_0^{E-1} \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_0^{E-1} \\
  \vdots \\
  \int_{T_N} \nabla u_{E-1} \cdot \nabla u_0^{E-1} \\
  \vdots \\
  \vdots \\
  \vdots \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_0^{E-1} \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_0^{E-1} \\
  \vdots \\
  \int_{T_1} \nabla u_{E} \cdot \nabla u_0^{E-1} \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_0^{E-1} \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_0^{E-1} \\
  \vdots \\
  \int_{T_N} \nabla u_{E} \cdot \nabla u_0^{E-1} \\
  \int_{T_1} \nabla u_{E+1} \cdot \nabla u_0^{E+1} \\
  \int_{T_1} \nabla u_{E+1} \cdot \nabla u_0^{E+1} \\
  \vdots \\
  \int_{T_1} \nabla u_{E+1} \cdot \nabla u_0^{E+1} \\
  \int_{T_N} \nabla u_{E+1} \cdot \nabla u_0^{E+1} \\
  \int_{T_N} \nabla u_{E+1} \cdot \nabla u_0^{E+1} \\
  \vdots \\
  \int_{T_N} \nabla u_{E+1} \cdot \nabla u_0^{E+1}
\end{bmatrix},$$

The matrix $S_{\gamma_0}$ is called the sensitivity matrix or Jacobian of the linearized EIT problem.

### 1.5.4 Quality of Sensitivity Matrix

“We will explain different data collection protocols in this section. Each protocol will generates a corresponding sensitivity matrix. We will apply the singular value decomposition explained in section xxx to the sensitivity matrix. Performance of the data collection protocol is closely related with the distribution of singular values. We will evaluate several sensitivity matrices from chosen data collection protocols. We will introduce the point spreading function and discuss a performance index of a chosen data collection method including the spatial resolution. We will mention how to design an EIT imaging experiment. This section will be completed later.”

### 1.6 Inverse Problem of EIT

We take $RC$ circuits as examples to introduce an inverse problem in EIT. After providing some intuitive understanding about the inverse problem in EIT, we formulate three EIT inverse problems including static imaging, time-difference imaging and frequency-difference imaging. Based on the observations in section 1.4.3, we study the ill-posedness in those inverse problems. Three inverse problems in EIT will be extensively studied in the following three chapters.
1.6.1 Inverse Problem of RC Circuit

We consider two simple examples of elementary inverse problems in RC circuits.

Example 1.6.1 Consider the series RC circuit. Injection current and measured voltage are \( I = I \angle 0 \) and \( V = V \angle \theta \), respectively, in their phasor forms. The inverse problem is to find the resistance \( R \) and the capacitance \( C \) from the relationship between \( I \) and \( V \).

Answer. From
\[
\frac{1}{Z} = \frac{1}{R + \frac{1}{i \omega C}} = \frac{V}{I} = V \angle \theta, \quad R = \frac{V \cos \theta}{I} \quad \text{and} \quad C = \frac{I}{\omega V \sin(-\theta)}.
\]
The number of unknowns is two and the number of measurements is also two including the real and imaginary parts of the impedance \( Z \).

Example 1.6.2 Repeat the above example for the parallel RC circuit.

Example 1.6.3 Consider the series RC circuit with two resistors and two capacitors. The inverse problem is to find \( R_1, R_2 \) and \( C_1, C_2 \) from the data \( I = I \angle 0 \) and \( V = V \angle \theta \).

Answer. From
\[
\frac{1}{Z} = \frac{1}{(R_1 + R_2) + \frac{1}{i \omega \left( \frac{1}{C_1} + \frac{1}{C_2} \right)}} = \frac{V}{I} = V \angle \theta, \quad R_1 + R_2 = \frac{V \cos \theta}{I} \quad \text{and} \quad C_1 + C_2 = \frac{I}{\omega V \sin(-\theta)}.
\]
The number of unknowns is four and the number of measurements is two including the real and imaginary parts of the impedance \( Z \). This results in infinitely many solutions.

The inverse problem in example 1.6.3 has no unique solution and is ill-posed in the sense of Hadamard explained in section xxx. Note that we may increase the number of measurements by separately measuring two voltages across \( R_1C_1 \) and \( R_2C_2 \) to uniquely determine \( R_1, C_1, R_2 \) and \( C_2 \). One may think of numerous RC circuits with multiple measurements that are either well-posed or ill-posed.

1.6.2 Formulation of EIT Inverse Problem

We assume an EIT system using \( E \) electrodes \( E_j \) for \( j = 1, 2, \cdots, E \). The admittivity inside an imaging domain \( \Omega \) at time \( t \), angular frequency \( \omega \) and position \( r \) is denoted as \( \gamma_{t, \omega}(r) = \sigma_{t, \omega}(r) + i \omega \epsilon_{t, \omega}(r) \).
**Static Imaging**

Static imaging in EIT is to produce an image of the admittivity \( \gamma_{t,\omega} \) from the NtD data \( F[\gamma_{t,\omega}] \) in (1.40). The image reconstruction requires inversion of the map

\[
\gamma_{t,\omega} \rightarrow F[\gamma_{t,\omega}]
\]

for a fixed time \( t \) and frequency \( \omega \). We may display images of \( \sigma_{t,\omega} \) and \( \omega\varepsilon_{t,\omega} \) separately. In each image, a pixel value is either \( \sigma_{t,\omega} \) or \( \omega\varepsilon_{t,\omega} \) in S/m. This kind of image is ideal for all applications since it provides absolute quantitative information. One may conduct multi-frequency static imaging by obtaining multiple NtD data sets at the same time at multiple frequencies. We may call this spectroscopic imaging. We may consecutively perform a series of static image reconstructions at multiple times to provide a time-series of admittivity images. Since static imaging is technically difficult in practice, we consider difference imaging methods.

**Time-difference Imaging**

Time-difference imaging produces an image of any difference \( \gamma_{t_2,\omega} - \gamma_{t_1,\omega} \) between two different times from the difference in corresponding two NtD data sets \( F[\gamma_{t_2,\omega}] - F[\gamma_{t_1,\omega}] \).

For single-frequency time-difference imaging, \( \omega \) is fixed. One may also perform multi-frequency time-difference imaging. Time-difference imaging is desirable for functional imaging of monitoring physiological events over time. Though it does not provide absolute values of \( \sigma_{t,\omega} \) and \( \omega\varepsilon_{t,\omega} \), it is more feasible in practice for applications where a reference NtD data at some time is available.

**Frequency-difference Imaging**

For applications where a time-referenced NtD data set is not available, we may consider frequency-difference imaging. It produces an image of any difference between \( \gamma_{t,\omega_2} \) and \( \gamma_{t,\omega_1} \) using NtD data sets \( F[\gamma_{t,\omega_2}] \) and \( F[\gamma_{t,\omega_1}] \), which are acquired at the same time. One may perform frequency-difference imaging at multiple frequencies using \( F[\gamma_{t,\omega_n}] \), \( n = 1, \ldots, F \).

Frequency-difference imaging may classify pathological conditions of tissues without relying on any previous data. Consecutive reconstructions of frequency-difference images at multiple times may provide functional information related with changes over time.

1.6.3 Ill-posedness of EIT Inverse Problem

Before we study three inverse problems in detail, we investigate their ill-posed characteristics based on the description in section 1.4.3 where we assumed that \( \gamma = \sigma \) for simplicity. For an injection current \( g \), we are provided with a limited number of voltage using a finite number of electrodes. The voltage data vector \( f \) corresponds to measured boundary voltages on portions of \( \partial\Omega \) where voltage-sensing electrodes are attached. The inverse problem is to determine the conductivity vector \( \sigma \), which is equivalent to obtain \( \bar{h}(\sigma) \), from several measurements of current-to-voltage pairs \( (g^m, f^m) \) for \( m = 1, \ldots, P \) where \( P \) is the number of projections.

The ill-posedness of the EIT inverse problem is related with the fact that reconstruction of \( \bar{h}(\sigma) \) from \( (g^m, f^m) \) with \( m = 1, \ldots, P \) is exponentially difficult as the size of \( \bar{h}(\sigma) \)
increases. This means that the ill-posedness gets worse as we increase the number of pixels for better spatial resolution. According to (1.44), the voltage at each pixel inside the imaging domain can be expressed as the weighted average of its neighboring voltages where weights are determined by the conductivity distribution. As explained in section 1.4.3, the measured voltage data vector $f$ is nonlinearly entangled in the global structure of the conductivity distribution. Any internal conductivity value $\sigma_k$ influences little on the boundary measurements $f$ especially when the position of $\sigma_k$ is away from positions of voltage-sensing electrodes. Figure 1.8 depict these phenomena, from which the ill-posedness is originated.

EIT reveals technical difficulties in producing high-resolution images due to the inherent insensitivity and nonlinearity. For a given finite number of electrodes, the amount of measurable information is also limited. Increasing the size of $\mathcal{A}(\sigma)$ for better spatial resolution makes the problem more ill-posed. To supply more measurements, we will have to increase the number of electrodes. With reduced gaps among electrodes, measured voltage differences will be smaller deteriorating SNRs. Beyond a certain spatial resolution or the pixel size, all the efforts to reduce the pixel size using a larger $\mathcal{A}(\sigma)$ result in poorer images since the severe ill-posedness takes over the benefit of additional information from the increased number of electrodes.

We should not expect EIT images to have a high spatial resolution needed for structural imaging. EIT cannot compete against X-ray CT or MRI in terms of the spatial resolution. One should find clinical significance of biomedical EIT from the fact that it provides unique new contrast information with a high temporal resolution using a portable machine.

1.7 Static Imaging

1.7.1 Mathematical Inverse Conductivity Problem

“We will summarize mathematical topics of inverse conductivity problem. Brief descriptions about Appendix A, B and C will be provided. This section will be completed later.”

1.7.2 Iterative Data Fitting Method

Most static image reconstruction algorithms for an $E$-channel EIT system can be viewed as a data fitting method as illustrated in Fig. 1.9. We first construct a computer model of an imaging object based on (1.37). With the discretization of the imaging domain into $N$ pixels as explained in section 1.5, we can express $\gamma$ as an admittivity vector

$$\gamma = [\gamma_1, \gamma_2, \cdots, \gamma_N]^T.$$  

Since we do not know the true admittivity $\gamma^*$ of the object, we assume an initial admittivity distribution $\gamma^m$ with $m = 0$ for the model. When we inject currents into both the object and the model, the corresponding measured and computed boundary voltages are different since $\gamma^m \neq \gamma^*$ in general. An image reconstruction algorithm iteratively updates $\gamma^m$ until it minimizes the difference between measured and computed boundary voltages.

To illustrate this idea, we define the following minimization problem:

$$\gamma^h = \arg \min_{\gamma \in \mathcal{A}} \Phi(\gamma), \quad \Phi(\gamma) := \left[ \frac{1}{2} \| F(\gamma^*) - F_C(\gamma) \|_2^2 \right]$$ (1.53)
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where ‘arg min’ is an operator which gives an energy functional minimizer, $\mathbb{F}(\gamma^*)$ is a measured NtD data vector, $\mathbb{F}_C(\gamma)$ is the computed NtD data vector and $\mathcal{A}$ is an admissible class for the admittivity. For the solution of (1.53), we may use an iterative nonlinear minimization algorithm such as the Newton-Raphson method.

In every iteration, we compute the sensitivity matrix or Jacobian $S_{\gamma m}$ in (1.52) by solving (1.37) with $\gamma m$ in place of $\gamma$. Solving the following linear equation

$$S_{\gamma m} \delta \gamma^m = \delta \mathbb{F}^m = \mathbb{F}(\gamma^*) - \mathbb{F}_C(\gamma^m)$$

for $\delta \gamma^m$ by

$$\delta \gamma^m = (S_{\gamma m} T S_{\gamma m})^{-1} S_{\gamma m} T \delta \mathbb{F}^m,$$

we update $\gamma^m$ as

$$\gamma^{m+1} = \gamma^m + \delta \gamma^m.$$ We may stop when

$$\|\delta \gamma^m\| < \delta$$

where $\delta$ is a tolerance.

### 1.7.3 Static Imaging using 4-channel EIT System

In order to understand the algorithm in (1.53) clearly, we consider a simple example using a 4-channel EIT system. We inject sinusoidal current $i_j(t) = I \cos \omega t$ to each electrode pair $E_j$ and $E_{j+1}$ for $j = 1, 2, 3$ and 4 with $E_5 = E_1$. From these 4 projections, we acquire 16 voltages:

$$F(\gamma^*) = \begin{pmatrix} V^{1,1} \\ V^{1,2} \\ \vdots \\ V^{4,4} \end{pmatrix} \in \mathbb{C}^{16}.$$
Figure 1.10  NtD data from a 4-channel EIT system. “This figure will be replaced by new one.”

Figure 1.11  Discretized imaging domain for a 4-channel EIT system. “This figure will be replaced by new one.”

Table 1.3  NtD data from a 4-channel EIT system

<table>
<thead>
<tr>
<th></th>
<th>$V^{1,k}$</th>
<th>$V^{2,k}$</th>
<th>$V^{3,k}$</th>
<th>$V^{4,k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 1$</td>
<td>3.1456</td>
<td>-1.5555</td>
<td>-0.135</td>
<td>-1.4551</td>
</tr>
<tr>
<td>$k = 2$</td>
<td>-1.5555</td>
<td>2.9714</td>
<td>-1.3183</td>
<td>0.0977</td>
</tr>
<tr>
<td>$k = 3$</td>
<td>-0.135</td>
<td>-1.3183</td>
<td>2.7767</td>
<td>-1.3234</td>
</tr>
<tr>
<td>$k = 4$</td>
<td>-1.4551</td>
<td>-0.0977</td>
<td>-1.3234</td>
<td>2.8761</td>
</tr>
</tbody>
</table>

Figure 1.10 shows a circular imaging object $\Omega$, $\Re\{V^{1,1}e^{j\omega t}\}$, $\Re\{V^{1,2}e^{j\omega t}\}$, $\Re\{V^{1,3}e^{j\omega t}\}$ and $\Re\{V^{1,4}e^{j\omega t}\}$.

We divide the imaging domain as $\Omega = T_1 \cup T_2 \cup T_3 \cup T_4$ in Fig. 1.11. Assume that $\gamma$ is constant on each $T_j$ for $j = 2, 3, 4$ and $\gamma = 1$ on $T_1$. The goal is to recover $\gamma$ from the NtD data in Table 1.3 using the following iteration process.

1. Let $\gamma^0 = 1$ as the initial guess.

2. For each $\gamma^m = (\gamma_1^m, \gamma_2^m, \gamma_3^m, \gamma_4^m)^T$, $m = 1, 2, \ldots$, solve the forward problem of (1.37) with $\gamma = \gamma^m$ and get $u^j = u_j^m$. Figure 1.12 shows distributions of $u_j^m$ for $j = 1, 2, 3$ and 4.
3. Compute the sensitivity matrix $S_{\gamma m}$ in (1.52) as

$$S_{\gamma m} = \begin{pmatrix}
\int_{T_1} \nabla u_1 \cdot \nabla u_1 \\
\int_{T_1} \nabla u_1 \cdot \nabla u_2 \\
\vdots \\
\int_{T_1} \nabla u_4 \cdot \nabla u_4 \\
\int_{T_2} \nabla u_1 \cdot \nabla u_1 \\
\int_{T_2} \nabla u_1 \cdot \nabla u_2 \\
\vdots \\
\int_{T_2} \nabla u_4 \cdot \nabla u_4 \\
\int_{T_3} \nabla u_1 \cdot \nabla u_1 \\
\int_{T_3} \nabla u_1 \cdot \nabla u_2 \\
\vdots \\
\int_{T_3} \nabla u_4 \cdot \nabla u_4 \\
\int_{T_4} \nabla u_1 \cdot \nabla u_1 \\
\int_{T_4} \nabla u_1 \cdot \nabla u_2 \\
\vdots \\
\int_{T_4} \nabla u_4 \cdot \nabla u_4 
\end{pmatrix}$$

and compute

$$F(\gamma^*) - F_C(\gamma^m) = \begin{pmatrix} V^{1,1} \\
V^{1,2} \\
\vdots \\
V^{4,4} \end{pmatrix} - \begin{pmatrix} V^{1,1}(\gamma^m) \\
V^{1,2}(\gamma^m) \\
\vdots \\
V^{4,4}(\gamma^m) \end{pmatrix}.$$

4. Calculate $\delta \gamma = [\delta \gamma_1, \delta \gamma_2, \delta \gamma_3]^T$ by solving

$$S_{\gamma m} \delta \gamma^m = \delta F^m = F(\gamma^*) - F_C(\gamma^m).$$

5. Update $\gamma^{m+1} = \gamma^m + \delta \gamma^m$.

6. Repeat the process 2, 3, 4 and 5 until $\|\delta \gamma^m\|$ is smaller than a predetermined tolerance.

In step 4, we used $S_{\gamma m} \delta \gamma^m = F(\gamma^*) - F_C(\gamma^m)$ to update $\gamma^m$. Recall that solving the minimization problem of $\Phi(\gamma)$ with the 4-channel EIT is to find a minimizing sequence $\gamma^m$ such that $\Phi(\gamma^m)$ approaches to its minimum effectively. The reason for this choice is that the $\delta \gamma^m$ in step 4 makes $\Phi(\gamma + \delta \gamma^m) - \Phi(\gamma)$ smallest with a given unit norm of $\|\delta \gamma^m\|$.

To see this rigorously, assume that the true conductivity is $\gamma^*$ and the measured data is exact so that $V^{j,k} = V^{j,k}[\gamma^*]$. According to (1.49),

$$\Phi(\gamma) = \sum_{j,k=1}^4 \left| V^{j,k} - V^{j,k}[\gamma] \right|^2 = \sum_{j,k=1}^4 \left| \int_{\partial \Omega} [u_{\gamma,j} - u_{\gamma,k}] \frac{\partial u_{\gamma,k}}{\partial n} \right|^2.$$

Computation of the Frechét derivative of the functional $\Phi(\gamma)$ requires to investigate the linear change $\delta u := u_{\gamma+\delta \gamma} - u_{\gamma}$ subject to a small conductivity perturbation $\delta \gamma$. Note
that $\Phi(\gamma + \delta \gamma) \approx \Phi(\gamma) + D \Phi(\gamma)(\delta \gamma) + \frac{1}{2}D^2 \Phi(\gamma)(\delta \gamma, \delta \gamma)$. For simplicity, we assume that $\delta \gamma = 0$ near $\partial \Omega$. The relationship between $\delta \gamma$ and the linear change $\delta u$ can be explained by
\[
\left\{ \begin{array}{l}
\nabla \cdot (\delta \gamma \nabla u) \approx -\nabla \cdot (\gamma \nabla \delta u) \text{ in } \Omega \\
\frac{\partial (\delta u)}{\partial n}|_{\partial \Omega} = 0.
\end{array} \right.
\]
We have the following approximation:
\[
\int_\Omega \delta \gamma \nabla u^j \cdot \nabla u^k dx \approx V^{j,k}[\gamma + \delta \gamma] - V^{j,k}[\gamma].
\]

We want to find the direction $\delta \gamma$ which makes $\Phi(\gamma + \delta \gamma) - \Phi(\gamma)$ smallest with a given unit norm of $\|\delta \gamma\|$. The steepest descent direction $\delta \gamma = (\delta \gamma_1, \delta \gamma_2, \delta \gamma_3)^T$ can be calculated by solving the matrix equation:
\[
S_\gamma \delta \gamma = f(\gamma) - f_{\text{meas}}.
\]

To understand this, we recall that $\Phi(\gamma) = \frac{1}{2} \sum_{j,k=1}^4 |V^{j,k}[\gamma] - V^{j,k}|^2$ and
\[
\Phi(\gamma + \delta \gamma) - \Phi(\gamma) \approx \sum_{j,k=1}^4 \Re \left\{ (V^{j,k}[\gamma] - V^{j,k}) \left( (V^{j,k}[\gamma + \delta \gamma] - V^{j,k}[\gamma]) \right) \right\}.
\]
As the direction $\delta \gamma$ which makes $\Phi(\gamma + \delta \gamma) - \Phi(\gamma)$ smallest with a given norm $\|\delta \gamma\|$, we must choose $\delta \gamma$ such that
\[
V^{j,k}[\gamma + \delta \gamma] - V^{j,k}[\gamma] = V^{j,k} - V^{j,k}[\gamma].
\]
Due to
\[
\int_\Omega \delta \gamma \nabla u^j \cdot \nabla u^k dx \approx V^{j,k}[\gamma + \delta \gamma] - V^{j,k}[\gamma],
\]
the steepest descent direction $\delta \gamma$ must satisfy
\[
\int_\Omega \delta \gamma \nabla u^j_\gamma \cdot \nabla u^k dx = V^{j,k} - V^{j,k}[\gamma] \quad (j, k = 1, 2, 3, 4).
\]

1.7.4 Regularization

Since the Jacobian matrix in (1.54) is ill-conditioned as explained in section 1.5, we often use a regularization method. Using the Tikhonov type regularization, we set
\[
\gamma^2 = \arg \min_{\gamma \in A} [\Phi(\gamma) + \lambda \eta(\gamma)] \tag{1.58}
\]
where $\lambda$ is a regularization parameter and $\eta(\gamma)$ is a function measuring a regularity of $\gamma$. This results in the following update equation for the $n$th iteration:
\[
\delta \gamma^m = \left( S_{\gamma}^T S_{\gamma} + \lambda R \right)^{-1} S_{\gamma}^T F^m \tag{1.59}
\]
where $R$ is a regularization matrix.
This kind of method was first introduced in EIT by Yorkey et al. followed by numerous variations and improvements (Yorkey et al. 1987, Cheney et al. 1990, Cheney et al. 1999, Lionheart 2005). These include utilization of a priori information, statistical information, various forms of regularity conditions, adaptive mesh refinement and so on. Though this iterative approach is widely adopted for static imaging, it requires a large amount of computation time and produce static images with a low spatial resolution and poor accuracy for the reasons discussed in the next section. Beyond this classical technique in static imaging, new idea is in demand for better image quality.

“We will add more materials about the regularization.”

1.7.5 Technical Difficulty of Static Imaging

“We will improve the readability of this section.”

In a static EIT imaging method, we construct a forward model of the imaging object with a presumed admittivity distribution. Injecting the same currents into the model as the ones used in measurements, boundary voltages are computed to numerically simulate measured data. Since the initially guessed admittivity distribution is in general different from the unknown admittivity distribution of the object, there exist some differences between measured and computed voltages. Most static EIT imaging methods are based on a minimization technique where a sum of these voltage differences is minimized by adjusting the admittivity distribution of the model (Yorkey et al. 1987, Cheney et al. 1990, Woo et al. 1993, Lionheart et al. 2005, Adler and Lionheart 2006). Other methods may include layer stripping (XXXX) and D-bar (Siltanen et al. 2001) algorithms.

In order for a static EIT image reconstruction algorithm to be reliable, we should be able to construct a forward model that mimics every aspect of the imaging object except the internal Admittivity distribution. This requires knowledge of the boundary geometry, electrode positions and other sources of systematic artifacts in measured data. In practice, it is very difficult to obtain such information within a reasonable accuracy and cost and most static EIT image reconstruction algorithms are very sensitive to these errors. In this section, we analyze this technical difficulty.

When we inject current through a pair of electrodes \( E_j \) and \( E_{j+1} \), the induced voltage \( u^j_{\gamma,\Omega} \) is dictated by the applied Neumann data \( g^j \) of the injection current, geometry of the domain \( \Omega \) and \( \gamma \). That is, \( u^j_{\gamma,\Omega} \) satisfies approximately

\[
\nabla \cdot (\gamma \nabla u^j_{\gamma,\Omega}) = 0 \quad \text{in } \Omega, \quad \gamma \nabla u^j_{\gamma,\Omega} \cdot n = g^j \quad \text{on } \partial \Omega \quad (1.60)
\]

where \( g^j \) represents the Neumann data in (1.37).

Taking account of the nonlinear ill-posedness in EIT, most image reconstruction methods for EIT use the assumption that \( \gamma \) is a perturbation of a known reference distribution \( \gamma^0 \) so that we can linearize the nonlinear problem. The inverse problem is to find \( \delta \gamma := \gamma - \gamma^0 \) from the integral equation

\[
\int_{\Omega} \delta \gamma \nabla u^j_{\gamma^0,\Omega} \cdot \nabla u^k_{\gamma,\Omega} dx = \int_{\partial \Omega} \left[ \Lambda_{\gamma^0,\Omega}(g^j) - \Lambda_{\gamma,\Omega}(g^j) \right] g^k dS \quad \text{for all } j, k \quad (1.61)
\]

where \( \Lambda_{\gamma,\Omega}(g^j) := u^j_{\gamma,\Omega}|_{\partial \Omega} \) and \( dS \) is the surface element ????. In practice, the value of the right side of (1.61) is the potential difference \( u^j \) between electrodes \( E_k \) and \( E_{k+1} \).
If the change $\delta \gamma$ is small, we can approximate
\[
\int_{\Omega} \delta \gamma \nabla u^{g, \Omega}_{\gamma, \omega} \cdot \nabla u^{k, \Omega}_{\gamma, \omega} dx \approx \int_{\Omega} \delta \gamma \nabla u^{g, \Omega}_{\gamma, \omega} \cdot \nabla u^{k, \Omega}_{\gamma, \omega} dx
\]
and (1.61) becomes
\[
S_{\gamma, \Omega}(\delta \gamma) = b_{\gamma, \Omega} - b_{\gamma, \Omega}
\]
where $S_{\gamma, \Omega}(\delta \gamma)$ and $b_{\gamma, \Omega}$ are $L \times L$ vectors having its $(j-1)L + k$ component $\int_{\Omega} \delta \gamma \nabla u^{g, \Omega}_{\gamma, \omega} \cdot \nabla u^{k, \Omega}_{\gamma, \omega} dx$ and $\int_{\Omega} \Lambda_{\gamma, \Omega}(g^j) g^k dS$, respectively. We may view $S_{\gamma, \Omega}(\cdot)$ as a linear operator acting on $\delta \gamma$ and its discretized version in terms of the admittivity distribution is called the sensitivity matrix.

To solve the inverse problem (1.62), we construct a forward model of the imaging object with a presumed reference admittivity $\tilde{\gamma}^0$:
\[
\nabla \cdot (\tilde{\gamma}^0 \nabla u^{\Omega, \Omega}_{\gamma, \omega} ) = 0 \quad \text{in} \; \Omega_c, \quad \tilde{\gamma}^0 \nabla u^{\Omega, \Omega}_{\gamma, \omega} \cdot \nu = \tilde{g}^j \text{ on } \partial \Omega_c
\]
where $\Omega_c \subset \mathbb{R}^3$ is a computational domain mimicking the geometry of the imaging subject. $\tilde{g}^j$ the Neumann data mimicking the applied current $g^j$ and $u^{\Omega, \Omega}_{\gamma, \omega}$ viewing as the internal potential induced by the current corresponding to the Neumann data $\tilde{g}^j$.

The forward model (1.63) is used to compute the reference boundary voltage $\Lambda_{\gamma, \Omega}(\tilde{g}^j) = u^{\Omega, \Omega}_{\gamma, \omega} |_{\partial \Omega_c}$ which is expected to be substituted for $\Lambda_{\gamma, \Omega}(g^j)$ in (1.62). If we have the exact forward modeling $S_{\gamma, \Omega}(\cdot) = S_{\gamma, \Omega}(\cdot)$ and $b_{\gamma, \Omega} = b_{\gamma, \Omega} = b_{\gamma, \Omega} - b_{\gamma, \Omega}$, we may obtain reasonably accurate images of $\delta \gamma$ by inverting the discretized version of the linear operator $S_{\gamma, \Omega}(\cdot)$ with the use of regularization. Knowing that we can not avoid forward modeling errors, a major drawback of static imaging is from the fact that the reconstruction problem (1.62) is very sensitive to geometric modeling errors in the computed reference data $\Lambda_{\gamma, \Omega}(\tilde{g}^j)$ including boundary geometry errors on $\Omega_c$ and electrode positioning errors on $\tilde{g}^j$. It would be very difficult to get accurate data $\Lambda_{\gamma, \Omega}(\tilde{g}^j)$ at a reasonable cost in practical environments.

In order to deal with undesirable effects of modeling errors, we investigate two difference imaging methods in the following sections. We expect that time or frequency derivatives of the NtD data $\Lambda_{\gamma, \omega}$ cancel out the effects of geometry errors on $\partial \Omega$.

### 1.8 Time-difference Imaging

In time-difference EIT (tdEIT), measured data at two different times are subtracted to produce images of changes in the complex conductivity distribution with respect to time. Since the data subtraction can effectively cancel out common errors, tdEIT has shown its potential as a new functional imaging modality in several clinical application areas. In this section, we consider multi-frequency time-difference EIT (mftdEIT) imaging. After formulating the mftdEIT imaging problem, we study the mftdEIT image reconstruction algorithm. Equivalent homogeneous admittivity of an imaging object will be introduced in the derivation of the algorithm.

#### 1.8.1 Time-difference Imaging Problem Summary

We assume an imaging object $\Omega$ bounded by its surface $\partial \Omega$. The isotropic admittivity in $\Omega$ at time $t$, angular frequency $\omega$ and position $r = (x, y, z)$ is denoted $\gamma_{t, \omega}(r) = \sigma_{t, \omega}(r) + \epsilon_{t, \omega}(r)$.
\[ i \omega \tau_{t,\omega}(r). \] Attaching surface electrodes \( E_j \) for \( j = 1, 2, \ldots, E \) on \( \partial \Omega \), we inject a sinusoidal current \( i(t) = I \cos(\omega t) \) between a chosen pair of electrodes. A distribution of voltage in \( \Omega \) is produced and we can express it as \( V_{t,\omega}(r) \cos(\omega t + \delta_{t,\omega}(r)) \).

Assuming an EIT system using \( E \) electrodes, we inject the \( j \)th current between the adjacent pair of electrodes denoted as \( E_j \) and \( E_{j+1} \) for \( j = 1, 2, \ldots, E \). The time-harmonic voltage subject to the \( j \)th injection current is denoted as \( u_{t,\omega}^j \) which is a solution of equation (1.37) with \( g \) replaced by \( g^j \). We assume that the EIT system is equipped with \( E \) voltmeters and each of them measures a boundary voltage between an adjacent pair of electrodes, \( E_k \) and \( E_{k+1} \) for \( k = 1, 2, \ldots, E \).

Using an mfdEIT system, we collect complex boundary voltage data at multiple frequencies for a certain period of time. Assuming that we collected \( E^2 \) number of complex boundary voltage data at each sampling time \( t \) and frequency \( \omega \), we can express a complex boundary voltage vector as \( \mathbf{V}_{t,\omega} \) for \( \omega = \omega_1, \omega_2, \ldots, \omega_F \). We assume that the EIT system is equipped with \( E \) voltmeters and each of them measures a boundary voltage between an adjacent pair of electrodes, \( E_k \) and \( E_{k+1} \) for \( k = 1, 2, \ldots, E \).

For \( t = t_1, t_2, \ldots, t_N \) and \( \omega = \omega_1, \omega_2, \ldots, \omega_F \), we are provided with \( N \) data vectors for each one of \( F \) frequencies. To perform tdEIT imaging, we should use the following complex boundary voltage data vectors at a reference time \( t_0 \): Alternatively, we may adopt a column vector representation as

\[
\mathbf{V}_{t_0,\omega} = \left[ V_{t_0,\omega}^{1,1} \cdots V_{t_0,\omega}^{1,E} \ V_{t_0,\omega}^{2,1} \cdots V_{t_0,\omega}^{2,E} \ \cdots \ V_{t_0,\omega}^{E,1} \cdots V_{t_0,\omega}^{E,E} \right]^T. \tag{1.64}
\]

For \( t = t_1, t_2, \ldots, t_N \) and \( \omega = \omega_1, \omega_2, \ldots, \omega_F \), we are provided with \( N \) data vectors for each one of \( F \) frequencies. To perform tdEIT imaging, we should use the following complex boundary voltage data vectors at a reference time \( t_0 \): Alternatively, we may adopt a column vector representation as

\[
\mathbf{V}_{t_0,\omega} = \left[ \mathbf{V}_{t_0,\omega}^{1,1} \cdots \mathbf{V}_{t_0,\omega}^{1,E} \mathbf{V}_{t_0,\omega}^{2,1} \cdots \mathbf{V}_{t_0,\omega}^{2,E} \ \cdots \mathbf{V}_{t_0,\omega}^{E,1} \cdots \mathbf{V}_{t_0,\omega}^{E,E} \right]^T. \tag{1.65}
\]

for \( \omega = \omega_1, \omega_2, \ldots, \omega_F \). The mfdEIT imaging problem is to produce time series of difference images using for \( t = t_1, t_2, \ldots, t_N \) at each one of \( \omega = \omega_1, \omega_2, \ldots, \omega_F \).

### 1.8.2 Equivalent Homogeneous Admittivity

For a given admittivity distribution \( \gamma_{t,\omega} \), we define the equivalent homogeneous admittivity \( \tilde{\gamma}_{t,\omega} \) as a complex number that minimizes

\[
\sum_{j=1}^{E} \int_{\Omega} \left| \gamma_{t,\omega}(\mathbf{r}) \nabla u_{t,\omega}^j(\mathbf{r}) - \tilde{\gamma}_{t,\omega} \nabla \tilde{u}_{t,\omega}^j(\mathbf{r}) \right|^2 d\mathbf{r} + \beta \int_{\Omega} \left| \gamma_{t,\omega}(\mathbf{r}) - \tilde{\gamma}_{t,\omega} \right|^2 d\mathbf{r}
\]

where \( \tilde{u}_{t,\omega}^j \) is the voltage satisfying equation (1.37) with \( \tilde{\gamma}_{t,\omega} \) in place of \( \gamma_{t,\omega} \) and \( \beta \) is a weighting constant. We assume that \( \gamma_{t,\omega} \) is a small perturbation of \( \tilde{\gamma}_{t,\omega} \).

We set a reference frequency \( \omega_0 \) as well as the reference time \( t_0 \). We assume that the complex boundary voltage vector \( \mathbf{V}_{t_0,\omega_0} \) is available at \( t = t_0 \) and \( \omega = \omega_0 \). Defining

\[
\alpha_{t,\omega} := \frac{1}{E^2} \sum_{j,k=1}^{E} \frac{V_{t_0,\omega_0}^{j,k}}{V_{t_0,\omega_0}^{j,k}},
\]

 Electrical Impedance Tomography
Thus, for all
\[ p \in \mathbb{C}^{2} \] roughly because
\[
\alpha_{t, \omega} = \frac{1}{E^2} \sum_{j, k=1}^{E} \int_{\Omega} \frac{\gamma_{t, \omega} \nabla u_{t, \omega}^{j} \cdot \nabla u_{t, \omega}^{k} \, dr}{\int_{\Omega} \gamma_{t, \omega} \nabla u_{t, \omega}^{j} \cdot \nabla u_{t, \omega}^{k} \, dr}
\] (1.66)

we have the following approximation:
\[
\frac{1}{\gamma_{t, \omega}} = \frac{1}{\gamma_{t, \omega}} \approx \frac{\gamma_{t, \omega} - \gamma_{t, \omega}}{\gamma_{t, \omega} \gamma_{t, \omega}}.
\]

Hence, for all \( p = (k - 1) \times E + j \) with \( j, k = 1, 2, \cdots, E \),
\[
I \left( \mathbb{F}_{t, \omega} - \mathbb{F}_{t, \omega} \right) \cdot e_{p} = \int_{\Omega} \frac{\gamma_{t, \omega} - \gamma_{t, \omega}}{\gamma_{t, \omega}} \gamma_{t, \omega} \nabla u_{t, \omega}^{k} \cdot \nabla u_{t, \omega}^{j} \, dr
\] (1.67)

where \( u^{j} \) and \( u^{k} \) are solutions of equation (1.37) with \( \gamma_{t, \omega} = 1 \) for the \( j \)th and \( k \)th injection currents, respectively.

We now relate a time change of the complex boundary voltage with a time change of the internal complex conductivity. For \( p = (k - 1) \times E + j \) with \( j, k = 1, 2, \cdots, E \),
\[
I \left( \mathbb{F}_{t, \omega} - \mathbb{F}_{t, \omega} \right) \cdot e_{p} = \int_{\Omega} \frac{\gamma_{t, \omega} - \gamma_{t, \omega}}{\gamma_{t, \omega}} \gamma_{t, \omega} \nabla u_{t, \omega}^{k} \cdot \nabla u_{t, \omega}^{j} \, dr
\] (1.68)

Note that we utilized the reciprocity theorem section 1.4. Since we assumed that \( \gamma_{t, \omega} \) and \( \gamma_{t, \omega} \) are small perturbations of \( \hat{\gamma}_{t, \omega} \) and \( \hat{\gamma}_{t, \omega} \), respectively, we have the following approximation:
\[
\frac{1}{\gamma_{t, \omega}} \approx \frac{1}{\gamma_{t, \omega}} \approx \frac{\gamma_{t, \omega} - \gamma_{t, \omega}}{\gamma_{t, \omega} \gamma_{t, \omega}}.
\]
1.8.3 Linear Time-difference Algorithm using Sensitivity Matrix

We construct a computer model of the imaging object $\Omega$. Assume that the domain of the model is $\Lambda$ with its boundary $\partial \Lambda$. Discretizing the model into $Q$ elements or pixels as $\Lambda = \cup_{q=1}^{Q} \Lambda_q$, we define the time-difference image $g_{t,\omega}$ at time $t$ and frequency $\omega$ as

$$G_{t,\omega} = H_{t,\omega} - H_{t_0,\omega}$$ (1.69)

with

$$H_{t,\omega} = [\gamma_{t,\omega}^1 \gamma_{t,\omega}^2 \cdots \gamma_{t,\omega}^Q]^T$$ and

$$H_{t_0,\omega} = [\gamma_{t_0,\omega}^1 \gamma_{t_0,\omega}^2 \cdots \gamma_{t_0,\omega}^Q]^T$$

where $\gamma_{t,\omega}^q$ and $\gamma_{t_0,\omega}^q$ for $q = 1, 2, \cdots, Q$ are admittivity values of the imaging object at times $t$ and $t_0$, respectively, inside a local region corresponding to the $q$th pixel $\Lambda_q$ of the model $\Lambda$.

The model is assumed to be homogeneous with $\gamma_{t_0,\omega}^q = \gamma^0$ in $\Lambda$. Using $E$ electrodes, we inject current between the $j$th adjacent pair of electrodes to induce voltage $v^j$ in $\Lambda$.

We numerically solve equation (1.37) for $v^j$ by using the finite element method. We can formulate the sensitivity matrix $S_{\gamma^0} = [s_{pq}]$ in section 1.5 as

$$s_{pq} = \int_{\Lambda_q} \nabla v^k \cdot \nabla v^j d\mathbf{r} \quad \text{and} \quad p = (k - 1) \times E + j$$ (1.70)

for $j, k = 1, 2, \cdots, E$ and $q = 1, 2, \cdots, Q$. The maximal size of $S_{\gamma^0}$ is $E^2 \times Q$ and all of its elements are real numbers. Using the discretization and linearization, equation (1.68) becomes

$$F_{t,\omega} - F_{t_0,\omega} = \frac{1}{I_{\gamma_{t_0,\omega}}^2} \alpha_{t_0,\omega} \alpha_{t,\omega} S_{\gamma^0} (H_{t_0,\omega} - H_{t,\omega}).$$ (1.71)

Computing the truncated singular value decomposition (TSVD) of $S_{\gamma^0}$, we find $P \leq Q$ singular values that are not negligible. We can compute a pseudo-inverse matrix of $S$ after truncating its $(Q - P)$ negligible singular values. Denoting this inverse matrix as $\Lambda$, we have

$$G_{t,\omega} = H_{t,\omega} - H_{t_0,\omega} = -I_{\gamma_{t_0,\omega}}^2 \alpha_{t_0,\omega} \alpha_{t,\omega} \Lambda (F_{t,\omega} - F_{t_0,\omega}).$$ (1.72)

Note that $\Lambda$ is a real matrix whose maximal size is $Q \times E^2$. Since we do not know $\gamma_{t_0,\omega}$ in (1.72), we replace (1.72) by the following equation:

$$I_{t,\omega} = \frac{G_{t,\omega}}{\gamma_{t_0,\omega}} = R_{t,\omega} + i X_{t,\omega} = -i \alpha_{t_0,\omega} \alpha_{t,\omega} \Lambda (F_{t,\omega} - F_{t_0,\omega})$$ (1.73)

where $R_{t,\omega}$ and $X_{t,\omega}$ are real and imaginary parts of a reconstructed complex tdEIT image $I_{t,\omega}$, respectively.

We may reconstruct a time series of mfdEIT images $I_{t_n,\omega_f}$ for $f = 1, 2, \cdots, F$ at $n = 1, 2, \cdots, N$. Choosing $\omega_f$ at a low frequency below 1 kHz, we may assume that $\gamma_{t_0,\omega_f} = \sigma_{t_0,\omega_f}$, since we can neglect the effects of the permittivity at low frequencies. In such a case, (1.73) becomes

$$I_{t_n,\omega_f} = \frac{G_{t_n,\omega_f}}{\sigma_{t_0,\omega_f}} = R_{t_n,\omega_f} + i X_{t_n,\omega_f} = -i \alpha_{t_0,\omega_f} \alpha_{t_n,\omega_f} \Lambda (F_{t_n,\omega_f} - F_{t_0,\omega_f}).$$ (1.74)
Note that $I^\text{tn}_{t_0,\omega_f}$ in (1.74) has the same phase angle as $G^\text{tn}_{t_0,\omega_f}$.

The mftdEIT image reconstruction algorithm based on (1.74) produces both real- and imaginary-part tdEIT images at multiple frequencies. It provides a theoretical basis for proper interpretation of a reconstructed image using the equivalent homogeneous complex conductivity. From (1.74), we can see that real- and imaginary-part images represent $\sigma^t_{t_0,\omega} - \sigma^t_{t_0,\omega_0}$ and $\omega \epsilon^t_{t_0,\omega} - \omega \epsilon^t_{t_0,\omega_0}$, respectively. We can interpret them as fractional changes of $\sigma$ and $\omega \epsilon$ between times $t$ and $t_0$ with respect to the square of the equivalent homogeneous conductivity $\hat{\sigma}^2_{t_0,\omega_0}$ of time $t_0$ at a low frequency $\omega_0$.

We should note several precautions in using the mftdEIT image reconstruction algorithm of (1.74). First, since (1.67) is based on the reciprocity theorem, the EIT system must have a smallest possible reciprocity error. Second, the true admittivity distribution $\gamma^t_{t_0,\omega}$ inside the imaging object at time $t$ and $\omega$ should be a small perturbation of its equivalent homogenous admittivity $\hat{\gamma}^t_{t_0,\omega}$ in order for the approximations in (1.67) and (1.68) to be valid. This is the inherent limitation of the difference imaging method using the linearization. Third, the computed voltage $v$ in (1.70) may contain modeling errors. It would be desirable for the model $\Lambda$ of the imaging object $\Omega$ to have correct boundary shape and size. We may improve the model by incorporating a more realistic boundary shape in the three-dimension. Fourth, the number of non-negligible singular values of the sensitivity matrix should be maximized by optimizing the electrode configuration and data collection protocol. With these technical issues in mind, spectroscopic time-difference imaging with a high temporal resolution is promising for numerous applications described in section 1.11.

### 1.8.4 Other Time-difference Algorithms

“We will summarize other time-difference image reconstruction algorithms including the backprojection algorithm in Appendix D.”

### 1.9 Frequency-difference Imaging

Since tdEIT requires a time-referenced data, it is not applicable to cases where a single image in time is required or such a time-referenced data is not available. Examples include imaging of tumors (Soni et al 2004, Kulkarni et al 2008, Trokhanova et al 2008) and cerebral stroke (McEwan et al 2006, Romsauerova et al 2006 and 2007). Noting that admittivity spectra of numerous biological tissues show frequency-dependent changes (Geddes and Baker 1967, Gabriel et al 1996, Grimnes and Martinsen 2008, Oh et al 2008) frequency-difference EIT (fdEIT) has been proposed to produce images of changes in the admittivity distribution with respect to frequency. Lately, frequency-difference magnetic induction tomography (fdMIT) has been also suggested for the detection of cerebral stroke.

In early fdEIT methods, frequency-difference images were formed by back-projecting the logarithm of the ratio of two voltages at two frequencies (Griffiths and Ahmed 1987a and 1987b, Griffiths 1987, Griffiths and Zhang 1989, Fitzgerald et al 1999, Schlappa et al 2000). More recent studies adopted the sensitivity matrix with a voltage difference at two frequencies (Yerworth et al 2003, Romsauerova et al 2006 and 2007, Bujnowski and Wtorek 2007). In MIT, induced voltage is proportional to the square of the frequency. For this reason, one should scale the voltage at the second frequency by the square of the ratio of two frequencies before any subtraction. Brunner et al (2006) and Zolgharni et al (2009a and
2009b) adopted this frequency scaling in their fdMIT methods. All of these fdEIT and fdMIT methods are basically utilizing a simple voltage difference at two frequencies and a linearized image reconstruction algorithm. Alternatively, we may consider separately producing two static (absolute) images at two frequencies and then subtract one from the other as suggested by Zolgharni et al (2009a) for fdMIT. This approach, however, will not be able to alleviate the technical difficulties of the static imaging method.

Recently, Seo et al (2008) suggested a new fdEIT method using a weighted voltage difference at two frequencies. They proposed two different contrast mechanisms in a reconstructed frequency-difference image. The first is the contrast in admittivity values between an anomaly and background. The second is the frequency-dependence of an admittivity distribution to be imaged. Since the admittivity spectra of most biological tissues changes with frequency, we will assume an imaging object with a frequency-dependent background admittivity in the development of fdEIT theory. We will explain that a simple voltage difference between two frequencies should produce bigger artifacts when the background admittivity changes with frequency. After describing the reason why the weighted difference method is desirable, we will formulate an fdEIT image reconstruction algorithm using the concept of an equivalent homogeneous admittivity.

1.9.1 Frequency-difference Imaging Problem Summary

We assume the same setting as in section 1.8.1. Using an $E$-channel EIT system, we may inject $E$ number of currents through adjacent pairs of electrodes and measure the following voltage data set:

$$F_{t,\omega} = [V_{t,\omega}^{1,1}, \cdots, V_{t,\omega}^{1,E}, V_{t,\omega}^{2,1}, \cdots, V_{t,\omega}^{2,E}, \cdots, V_{t,\omega}^{E,1}, \cdots, V_{t,\omega}^{E,E}]^T.$$  \hspace{1cm} (1.75)

For $t = t_1, t_2, \cdots, t_N$ and $\omega = \omega_1, \omega_2, \cdots, \omega_F$, we are provided with $N$ data vectors for each one of $F$ frequencies. Let’s assume that we inject currents at two frequencies of $\omega_1$ and $\omega_2$ to obtain corresponding voltage data sets $F_{t,\omega_1}$ and $F_{t,\omega_2}$, respectively. The goal is to visualize changes of the admittivity distribution between $\omega_1$ and $\omega_2$ by using these two voltage data sets.

In tumor imaging or stroke detection using EIT, we are primarily interested in visualizing an anomaly from a background. This implies that we should reconstruct a local admittivity contrast. For a given injection current, however, the boundary voltage $F_{t,\omega}$ is significantly affected by the background admittivity, boundary geometry and electrodes positions, while the influence of a local admittivity contrast due to an anomaly is much smaller. Since we utilize two sets of boundary voltage data, $F_{t,\omega_1}$ and $F_{t,\omega_2}$ in fdEIT, we need to evaluate their capability to perceive the local admittivity contrast. As in tdEIT, the rationale is to eliminate numerous common errors by subtracting the background component of $F_{t,\omega_1}$ from $F_{t,\omega_2}$, while preserving the local admittivity contrast component.

1.9.2 Simple Difference ($F_{t,\omega_2} - F_{t,\omega_1}$)

The simple voltage difference $F_{t,\omega_2} - F_{t,\omega_1}$ may work well for an imaging object whose background admittivity does not change with frequency. A typical example is a saline phantom. For realistic cases where background admittivity distributions change
with frequency, it will produce artifacts in reconstructed fdEIT images. To understand this, let us consider a very simple case where the imaging object has a homogeneous admittivity distribution, that is, \( \gamma_{t,\omega} = \sigma_{t,\omega} + i \omega \epsilon_{t,\omega} \) is independent of the position. In such a homogeneous object, induced voltages \( \bar{u}_1 \) and \( \bar{u}_2 \) satisfy the Laplace equation with the same boundary data and the two corresponding voltage data vectors \( \bar{F}_1 \) and \( \bar{F}_2 \) are parallel in such a way that

\[
\bar{F}_2 = \gamma_{t,\omega} \bar{F}_1.
\]

When there exists a small anomaly inside the imaging object, we may assume that the induced voltages are close to the voltages without any anomaly. In other words, the voltage difference \( \bar{F}_2 - \bar{F}_1 \) at the presence of a small anomaly can be expressed as

\[
\bar{F}_2 - \bar{F}_1 \approx \bar{F}_1 = \gamma_{t,\omega} \bar{F}_1 = \beta \bar{F}_1
\]

for a complex constant \( \beta \). This means that the simple difference \( \bar{F}_2 - \bar{F}_1 \) significantly depends on the boundary geometry and electrode positions except the special case where \( \bar{F}_2 - \bar{F}_1 = 0 \). This is the main reason why the use of the simple difference \( \bar{F}_2 - \bar{F}_1 \) cannot deal with common modeling errors even for a homogeneous imaging object.

### 1.9.3 Weighted Difference \( \left( \bar{F}_{t,\omega_2} - \alpha \bar{F}_{t,\omega_1} \right) \)

An imaging object including a background and anomaly has an admittivity distribution \( \gamma_{t,\omega} \). We define a weighted difference of the admittivity at two different frequencies \( \omega_1 \) and \( \omega_2 \) at time \( t \) as

\[
\delta \gamma_{t,\omega_2} = \alpha \gamma_{t,\omega_2} - \gamma_{t,\omega_1}
\]

where \( \alpha \) is a complex number. We assume the following two conditions:

1. In the background region, especially near the boundary, \( \delta \gamma_{t,\omega_2} \approx 0 \).
2. In the anomaly, \( \delta \gamma_{t,\omega_2} \) is significantly different from 0.

In order to extract the anomaly from the background, we investigate the relationship between \( \bar{F}_{t,\omega_2} \) and \( \bar{F}_{t,\omega_1} \). We should find a way to eliminate the background influence while maintaining the information of the admittivity contrast across the anomaly. We decompose \( \bar{F}_{t,\omega_2} \) into a projection part onto \( \bar{F}_{t,\omega_1} \) and the remaining part:

\[
\bar{F}_{t,\omega_2} = \alpha \bar{F}_{t,\omega_1} + \bar{E}_{t,\omega_2}, \quad \alpha = \frac{\langle \bar{F}_{t,\omega_2}, \bar{F}_{t,\omega_1} \rangle}{\langle \bar{F}_{t,\omega_1}, \bar{F}_{t,\omega_1} \rangle}
\]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product of two vectors. Note that \( \bar{E}_{t,\omega_2} \) is orthogonal to \( \bar{F}_{t,\omega_1} \).

In the absence of the anomaly, we may set \( \gamma_{\omega_2} = \frac{1}{\alpha} \gamma_{\omega_1} \) and this results in \( \bar{F}_{t,\omega_2} = \alpha \bar{F}_{t,\omega_1} \). The projection term \( \alpha \bar{F}_{t,\omega_1} \) mostly contains the background information, while the orthogonal term \( \bar{E}_{t,\omega_2} \) holds the anomaly information. To be precise, \( \alpha \bar{F}_{t,\omega_1} \) provides the same information as \( \bar{F}_{t,\omega_1} \) which includes influences of the background admittivity, boundary geometry and electrode positions. The orthogonal term \( \bar{E}_{t,\omega_2} = \bar{F}_{t,\omega_2} - \alpha \bar{F}_{t,\omega_1} \) contains the core information about a nonlinear change due to the admittivity contrast across the anomaly. This explains why the weighted difference \( \bar{F}_{t,\omega_2} - \alpha \bar{F}_{t,\omega_1} \) must be used in fdEIT.
1.9.4 Linear Frequency-difference Algorithm using Sensitivity Matrix

In this section, we drop the time index $t$ assuming a same time for all cases. Applying a linear approximation in section 1.5, we get the following relation:

$$\int_{\Omega} \delta \gamma^2_{\omega_1}(r) \nabla u_{\omega_1}^j(r) \cdot \nabla u_{\omega_2}^k(r) \, dr \approx I \left( V_{\omega_2}^{j,k} - \alpha V_{\omega_1}^{j,k} \right), \quad j, k = 1, 2, \ldots, E. \quad (1.78)$$

Given $\alpha$, we can reconstruct an image of $\delta \gamma^2_{\omega_1}$ using the weighted difference $V_{\omega_2}^{j,k} - \alpha V_{\omega_1}^{j,k}$. Since $\alpha$ is not known in practice, we need to estimate it from $V_{\omega_2}^{j,k}$ and $V_{\omega_1}^{j,k}$ using (1.77).

We discretize the imaging object $\Omega$ as $\Omega = \bigcup_{i=1}^{N} \Omega_i$ where $\Omega_i$ is the $i$th pixel. Let $\chi_{\Omega_i}$ be the characteristic function of the $i$th element $\Omega_i$, that is, $\chi_{\Omega_i} = 1$ in $\Omega_i$ and zero otherwise. Let $\xi_1, \cdots, \xi_N$ be complex numbers such that $\sum_{i=1}^{N} \xi_i \chi_{\Omega_i}$ approximates $\sum_{i=1}^{N} \delta \gamma_{\omega_1} \chi_{\Omega_i} \approx \sum_{i=1}^{N} \xi_{\omega_1} \chi_{\Omega_i}$. By approximating $\nabla u_{\omega_2}^j \approx \frac{1}{\gamma_{\omega}} \nabla U^j$ where $U^j$ is the solution of (1.26) with $\gamma_{\omega} = 1$, it follows from (1.78) that

$$\sum_{i=1}^{N} \left( \xi_i \int_{\Omega_i} \nabla U^j(r) \cdot \nabla U^k(r) \, dr \right) \approx \int_{\Omega} \frac{\delta \gamma^2_{\omega_1}}{\gamma_{\omega_1}} \nabla U^j(r) \cdot \nabla U^k(r) \, dr \approx \left( V_{\omega_2}^{j,k} - \alpha V_{\omega_1}^{j,k} \right), \quad j, k = 1, 2, \ldots, E. \quad (1.79)$$

The reconstruction method using the approximation (1.78) is reduced to reconstruct $\sum_{i=1}^{N} \xi_i \chi_{\Omega_i}$ that minimizes the following:

$$\sum_{j,k=1}^{N} \left| \left( \xi_i \int_{\Omega_i} \nabla U^j(r) \cdot \nabla U^k(r) \, dr - (V_{\omega_2}^{j,k} - \alpha V_{\omega_1}^{j,k}) \right) \right|^2 \quad (1.80)$$

where $\alpha$ is the complex number described in section 1.9.3. In order to find $\xi = (\xi_1, \cdots, \xi_N)$, we use the sensitivity matrix $S\alpha = 0$ in (1.70). We can compute $\xi = (\xi_1, \cdots, \xi_N)$ by solving the following linear system through a truncated singular value decomposition (TSVD):

$$S\alpha, \xi = F_{\omega_2} - \alpha F_{\omega_1}. \quad (1.81)$$

It remains to compute the fdEIT image $\delta \gamma^2_{\omega_1}$ from the knowledge of $\xi$. We need to estimate the equivalent homogeneous (constant) admittivity $\hat{\gamma}_{\omega}$ corresponding to $\gamma_{\omega}$ to use the following approximation

$$\delta \gamma^2_{\omega_1} \approx \frac{\gamma_{\omega}}{\hat{\gamma}_{\omega}} \sum_{i=1}^{N} \xi_i \chi_{\Omega_i}. \quad (1.82)$$

From the divergence theorem, we obtain the following relation:

$$\frac{\hat{\gamma}_{\omega}}{\gamma_{\omega}} \frac{\nabla u_{\omega}^j \cdot \nabla w_{\omega}^j}{\nabla u_{\omega}^k \cdot \nabla w_{\omega}^k} \approx \frac{V_{\omega}^{j,k}}{V_{\omega}^{j,j}} \quad \text{for any } j, k \in \{1, 2, \cdots, E\}. \quad (1.83)$$

For an $E$-channel mfEIT system, we may choose

$$\frac{\hat{\gamma}_{\omega}}{\gamma_{\omega}} = \frac{1}{2E} \sum_{j=1}^{E} \left( \frac{V_{\omega}^{j,j+3}}{V_{\omega}^{j+3,j+3}} + \frac{V_{\omega}^{j,j-3}}{V_{\omega}^{j-3,j-3}} \right). \quad (1.84)$$
where we identify $E + j = j$ and $- j = E - j$ for $j = 1, 2, 3$.

We reconstruct an fdEIT image $\delta \gamma_{\omega_1}$ by

$$\delta \gamma_{\omega_1} = I \hat{\gamma}_{\omega_1} \hat{\gamma}_{\omega_2} A (F_{\omega_2} - \alpha F_{\omega_1}) = I \hat{\gamma}_{\omega_0} \hat{\gamma}_{\omega_2} A (F_{\omega_2} - \alpha F_{\omega_1})$$  \hspace{1cm} (1.82)$$

where $A$ is a pseudo-inverse of $S_{\gamma_0}$ estimated by the tSVD. In (1.82), $\hat{\gamma}_{\omega_1}$, $\hat{\gamma}_{\omega_2}$, $\hat{\gamma}_{\omega_0}$ can be estimated from (1.81) using another low frequency measurement $F_{\omega_0}$. If we choose $\omega_0$ low enough, $\hat{\gamma}_{\omega_0}$ may have a negligibly small imaginary part. In such a case, we may set $\delta \gamma_{\omega_1} / \hat{\gamma}_{\omega_0}$ as a reconstructed fdEIT image, which is equivalent to the complex image $\delta \gamma_{\omega_1}$ divided by an unknown real constant. In practice, it would be desirable to set $\omega_0$ smaller than 1 kHz, for example 100 Hz.

This scaling will be acceptable for applications where we are mainly looking for a contrast change within an fdEIT image. These may include detections of tumors and strokes. In order to quantitatively interpret absolute pixel values of an fdEIT image, we must estimate the value of $\hat{\gamma}_{\omega_0}$, which requires the knowledge of the object size, boundary shape and electrode positions. Alternatively, we may estimate values of $\hat{\gamma}_{\omega_1}$ and $\hat{\gamma}_{\omega_2}$ in (1.82) without using the third frequency $\omega_0$. This will again need geometrical information about the imaging object and electrode positions.

### 1.10 Imaging Experiment

“We will illustrate examples of static imaging, time-difference imaging and frequency-difference imaging. There will be many figures of imaging setups, measured data, sensitivity matrices and images. We will also summarize numerous experimental studies from literature. This section will be completed later.”

### 1.11 Applications

EIT has several merits such as its portability and high temporal resolution even though its spatial resolution is poor. Noting that common errors may canceled out by a data subtraction method, time-difference EIT imaging has shown its potential in clinical applications where monitoring temporal changes of a conductivity distribution is needed. Frequency-difference EIT imaging aims to detect an anomaly such as bleeding in the brain and tumor tissue in the breast.

“We will add more materials on applications. This section will be completed later.”

### 1.12 Uniqueness theory in EIT and direct reconstruction method(Optional)

In this subsection, EIT data will be the full NtD map $\Lambda$, defined in section. Calderón(1980) made the following observation which plays a key role in achieving the theoretical development of EIT, especially uniqueness theory.

For quick and easy explanation, we assume the followings throughout this section.

- $\Omega \subset \mathbb{R}^n$ with its $C^2$-boundary $\partial \Omega$. 

• \( \gamma \) is real and \( \gamma \in C_0^2(\Omega) \) with \( \gamma = 1 \) in \( \mathbb{R}^n \setminus \Omega \).

• \( q = \frac{\nabla \sqrt{\gamma}}{\sqrt{\gamma}} \) in \( \Omega \) and \( q = 0 \) in \( \mathbb{R}^n \setminus \Omega \).

• \( \gamma^0 = 1 \) is the background conductivity.

**Theorem 1.12.1** [Calderón’s Approach] Let \( \gamma^0 \in \mathbb{C} \) and let \( \delta \sigma, \delta \gamma \in C_0^2(\Omega) \). Denote \( \gamma_t(r) = \gamma^0 + t\delta \gamma(r) \), \( \tilde{\gamma}_t(r) = \gamma^0 + t\delta \gamma(r) \) for \( r \in \mathbb{R}^3 \).

If for any \( g \in H_{-1/2}^0(\partial \Omega) \)

\[
\frac{d}{dt} \Lambda_{\gamma_t}(g)|_{t=0} = \frac{d}{dt} \Lambda_{\gamma_0}(g)|_{t=0} \quad \text{on} \quad \partial \Omega
\]

then

\( \delta \gamma = \tilde{\delta} \gamma \) in \( \Omega \)

To prove the above theorem, Calderón used the set of special pairs of harmonic functions which is dense in \( L^1(\Omega) \).

**Lemma 1.12.2** If \( \xi, \eta \in \mathbb{R}^3 \) (or \( \mathbb{R}^2 \)) satisfy

\[
|\xi| = |\eta| \quad \text{and} \quad \xi \cdot \eta = 0,
\]

then both \( v = e^{r(\eta + i\xi)} \) and \( w = e^{-r(\eta - i\xi)} \) are harmonic in the entire space \( \mathbb{R}^3 \) (or \( \mathbb{R}^2 \)). Moreover,

\[
\{ \nabla v \xi \cdot \nabla w \xi : \xi \in \mathbb{R}^3 \} \quad \text{is dense in} \quad L^1(\Omega)
\]

where \( v_\xi = e^{r(\xi^* + i\xi)} \) and \( w_\xi = e^{-r(\xi - i\xi)} \) and \( \xi^* = \frac{|\xi|}{\sqrt{\xi_1^2 + \xi_2^2}} (-\xi_2, \xi_1, 0) \).

**Proof.** The \( v \) and \( w \) are harmonic because

\[
\nabla^2 v = \left[ (|\eta|^2 - |\xi|^2) + i2\xi \cdot \eta \right] v = 0 = \nabla^2 w
\]

Since \( \nabla v_\xi \cdot \nabla w_\xi = -2|\xi|^2 e^{2i\xi \cdot r} \),

\[
\{ \nabla v_\xi \cdot \nabla w_\xi : \xi \in \mathbb{R}^3 \} = \{ e^{2i\xi \cdot r} : \xi \in \mathbb{R}^3 \}
\]

which is clearly dense in \( L^1(\Omega) \) due to the Fourier representation formula.

**Proof.** [Proof of theorem 1.12.1.] For \( g \in H_{-1/2}^{1/2}(\partial \Omega) \), let \( u^\theta_t \) be a solution of

\[
\begin{cases}
\nabla \cdot (\gamma \nabla u^\theta_t) = 0 & \text{in} \ \Omega \\
-\gamma \nabla u^\theta_t \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \int_{\partial \Omega} u^\theta_t \ ds = 0
\end{cases}
\]

(1.85)

Taking the derivative of the problem (1.85) with respect to \( t \), we have

\[
\begin{cases}
\nabla \cdot (\frac{\partial}{\partial t} \gamma_0 \nabla u^\theta_t) = -\nabla \cdot (\gamma_0 \nabla \frac{\partial}{\partial t} u^\theta_t) & \text{in} \ \Omega \\
-\gamma \nabla (u^\theta_t) \cdot \mathbf{n}|_{\partial \Omega} = 0
\end{cases}
\]

(1.86)
Here, we use the assumption that $\delta \sigma |_{\partial \Omega} = 0$. By multiplying $u_i^\phi$ to (1.87) and integrating over $\Omega$, we have

$$\int_{\Omega} \delta \gamma u_i^\phi \cdot \nabla u_i^\phi \, dr = - \int_{\Omega} \gamma \nabla \frac{\partial}{\partial t} u_i^\phi \cdot \nabla u_i^\phi \, dr$$

$$= - \int_{\partial \Omega} \frac{d}{dt} \Lambda_{\gamma_t}(g) \phi \, ds$$

At $t = 0$, it becomes

$$\int_{\Omega} \delta \gamma u_i^0 \cdot \nabla u_i^0 \, dr = - \int_{\partial \Omega} \frac{d}{dt} \Lambda_{\gamma_t}(g)|_{t=0} \phi \, ds$$

We also have the same identity for $\tilde{\delta} \gamma$:

$$\int_{\Omega} \tilde{\delta} \gamma u_i^0 \cdot \nabla u_i^0 \, dr = - \int_{\partial \Omega} \frac{d}{dt} \Lambda_{\gamma_t}(g)|_{t=0} \phi \, ds.$$

Hence, it follows from the assumption $\Lambda_{\gamma_t}(g)|_{t=0} = \Lambda_{\gamma_t}(g)|_{t=0}$ that

$$\int_{\Omega} (\delta \gamma - \tilde{\delta} \gamma) \nabla u_i^0 \cdot \nabla u_i^0 \, dr, \quad \forall g, \phi \in H^{-\frac{1}{2}}(\partial \Omega) \quad (1.87)$$

It must be $\delta \gamma = \tilde{\delta} \gamma$ because

$$\{ \nabla v_\xi \cdot \nabla w_\xi : \xi \in \mathbb{R}^3 \} \subset \{ \nabla u_i^0 \cdot \nabla u_i^\phi : g, \phi \in H^{-\frac{1}{2}}(\partial \Omega) \}$$

and $\{ \nabla v_\xi \cdot \nabla w_\xi : \xi \in \mathbb{R}^3 \}$ is dense in $L^1(\Omega)$ from Lemma 1.12.2.

Let us begin with explaining the scattering transform that transform the conductivity equation $\nabla \cdot (\gamma \nabla u) = 0$ to the Schrödinger equation $(\nabla^2 + q) \psi = 0$. This transform was firstly used to prove the uniqueness of EIT for $\gamma \in C^{1,1}(\Omega)$ by Sylvester and Uhlmann (1987). The following lemma explain this scattering transform.

**Lemma 1.12.3** Let $\gamma \in C^2(\overline{\Omega})$ and let $u$ satisfy

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in} \ \Omega$$

Then $\psi := \sqrt{\gamma} u$ satisfies

$$-\nabla^2 \psi + \frac{\nabla^2 \sqrt{\gamma}}{\sqrt{\gamma}} \psi = 0 \quad (1.88)$$

**Proof.** The proof follows from the following direct computation:

$$0 = \nabla \cdot (\gamma \nabla u)$$

$$= \nabla \cdot (\sqrt{\gamma} \nabla (\sqrt{\gamma} u)) - \nabla \cdot (\sqrt{\gamma} u \nabla \sqrt{\gamma})$$

$$= \nabla \cdot (\sqrt{\gamma} \nabla \psi) - \nabla \cdot (\psi \nabla \sqrt{\gamma})$$

$$= \sqrt{\gamma} \nabla^2 \psi - \psi \nabla^2 \sqrt{\gamma}$$
Remark 1.12.1 From Lemma d-bar:lemma1, \( \psi = \sqrt{\gamma} \) is the solution of
\[
\begin{cases}
-\nabla^2 \psi + q \psi = 0 & \text{in } \Omega \\
\psi|_{\partial \Omega} = 1
\end{cases}
\quad \left( q = \frac{\nabla^2 \sqrt{\gamma}}{\sqrt{\gamma}} \right)
\tag{1.89}
\]
Nachman used this fact has been used to develop a 2-D constructive identification method of \( \gamma \) named \( \overline{\partial} \)-method.

1.12.1 Uniqueness and Three-dimensional Reconstruction from NtD Data: Infinite Measurements
In this section, we briefly explain some impressive results in the uniqueness question and 3-D reconstruction method in EIT mainly due to Sylvester-Uhmann(1987) and Nachman(1988). However, the reconstruction formula suggested in this section should not be used as an EIT algorithm in practical situation.

Define DtN map \( \Gamma_q : H^{-1/2}_0(\partial \Omega) \to H^{1/2}_0(\partial \Omega) \) by
\[
\Gamma_q(g) = u^1|_{\partial \Omega}
\]
where \( u_j \) satisfies
\[
\begin{cases}
L_{q_j} u_j := \nabla^2 u_j - q_j u_j = 0 & \text{in } \Omega, \\
\frac{\partial u_j}{\partial n}|_{\partial \Omega} = g
\end{cases}
\tag{1.90}
\]
The goal is to prove
\[
\Gamma_{q_1} = \Gamma_{q_2} \Rightarrow q_1 = q_2.
\]

Lemma 1.12.4 Assume \( \Gamma_{q_1} = \Gamma_{q_2} \). For any \( u_1 \) and \( u_2 \) satisfying \( \nabla^2 u_j - q_j u_j = 0 \) in \( \Omega \),
\[
\int_\Omega (q_2 - q_1) u_1 u_2 = 0.
\]

Proof. By the definition, we have
\[
\int_\Omega \nabla u_j \cdot \nabla \phi - q_j u_j \phi dx = \int_{\partial \Omega} \frac{\partial u_j}{\partial n} \phi ds.
\]
for any \( \phi \in H^1(\Omega) \). Hence,
\[
\int_\Omega \nabla u_1 \cdot \nabla u_2 - q_1 u_1 u_2 dx = \int_{\partial \Omega} \frac{\partial u_1}{\partial n} u_2 ds
\]
\[
\int_\Omega \nabla u_1 \cdot \nabla u_2 - q_2 u_1 u_2 dx = \int_{\partial \Omega} u_1 \frac{\partial u_2}{\partial n} ds.
\]
Subtracting the above two equations yields
\[
\int_\Omega (q_2 - q_1) u_1 u_2 dx = \int_{\partial \Omega} g(\Gamma_{q_2}(g) - \Gamma_{q_1}(g)) ds.
\]
It then follows that if \( \Gamma_{q_2} = \Gamma_{q_1} \), then
\[
\int_\Omega (q_2 - q_1) u_1 u_2 dx = 0.
\]
Theorem 1.12.6 The set \( \{ u_1, u_2 : u_j \in C^2(\Omega) \text{ satisfying } \nabla^2 u_j - q_j u_j = 0 \} \) is dense in \( L^1(\Omega) \). Hence, \( \Gamma_{q_1} = \Gamma_{q_2} \implies q_1 = q_2 \text{ in } \Omega \).

Proof. For each \( k \in \mathbb{R}^3 \) and \( r \in \mathbb{R}^+ \), we can choose \( \eta, \xi \in \mathbb{R}^3 \) (six unknowns) satisfying four equations:
\[
0 = \eta \cdot \xi = \eta \cdot k = k \cdot \xi, \quad |\eta|^2 = r^2 |\xi|^2 + |k|^2
\]
Denoting \( \zeta_1 := i\eta - (r\xi + k) \) and \( \zeta_2 := -i\eta - (-r\xi + k) \), we have \( \zeta_1 + \zeta_2 = -2k \) and \( \zeta_j \cdot \zeta_j = 0 \) \( (j = 1, 2) \). Let \( u_j \) be the solution given in Lemma 1.12.5 corresponding to \( q_j \), i.e.,
\[
u_j = e^{i\xi} \zeta_j[1 + \psi_j(x)]
\]
Then \( u_1 u_2 = e^{-2ik \cdot x} [1 + \psi_{\zeta_1} + \psi_{\zeta_2} + \psi_{\zeta_1} \psi_{\zeta_2}] \rightarrow e^{-2ik \cdot x} \) as \( r \rightarrow \infty \) in \( L^1(\Omega) \)-sense. Therefore, we have
\[
0 = \int_{\Omega} (q_2 - q_1) u_1 u_2 \rightarrow \int_{\Omega} (q_2 - q_1) e^{-2ik \cdot x} \text{ as } r \rightarrow \infty
\]
Since \( k \in \mathbb{R}^3 \) is arbitrary, we finally have \( q_1 = q_2 \).
The next observation provides an explicit representation formula for $q$ from the Knowledge of NtD map.

**Observation 1.12.1 (Nachman’s Reconstruction)** Let $u_\xi = e^{i\xi \cdot \psi}[1 + \psi_\xi(x)]$ in Lemma 1.12.3. Then

$$
\hat{q}(\xi) = \lim_{|\xi| \to \infty} \int_{\partial \Omega} e^{-ix(\xi + \zeta)} \left[ \frac{\partial u_\zeta}{\partial n} (x) + i(\xi + \zeta) \cdot n \Lambda \frac{\partial u_\zeta}{\partial n} (\partial \Omega)(x) \right] ds \quad (1.92)
$$

**Proof.** Since $e^{-ix(\xi + \zeta)}u_\zeta(x) \to 1$ as $|\xi| \to \infty$, we have

$$
\lim_{|\xi| \to \infty} \int_{\partial \Omega} e^{-ix(\xi + \zeta)}q(x)u_\zeta(x)dx = \int_{\partial \Omega} e^{-ix\xi}q(x)dx = \hat{q}(\xi). \quad (1.93)
$$

On the other hand, since $qu_\xi = \nabla^2 u_\zeta$, it follows from the divergence theorem that

$$
\int_{\Omega} e^{-ix(\xi + \zeta)}\nabla^2 u_\zeta(x)dx = \int_{\partial \Omega} e^{-ix(\xi + \zeta)} \left[ \frac{\partial u_\zeta}{\partial n} (x) + i(\xi + \zeta) \cdot n \Lambda \frac{\partial u_\zeta}{\partial n} (\partial \Omega)(x) \right] ds.
$$

Thus we have

$$
\hat{q}(\xi) = \lim_{|\xi| \to \infty} \int_{\partial \Omega} e^{-ix(\xi + \zeta)} \left[ \frac{\partial u_\zeta}{\partial n} (x) + i(\xi + \zeta) \cdot n \Lambda \frac{\partial u_\zeta}{\partial n} (\partial \Omega)(x) \right] ds. \quad (1.94)
$$

### 1.12.2 Nachmann’s D-bar Method in 2-D


**Overall idea of D-bar method.**

The D-bar method is based on the fact from Lemma 1.12.3: $\psi = \sqrt{\gamma}$ is the solution of

$$
\left\{ \begin{array}{l}
-\nabla^2 \psi + q\psi = 0 \quad \text{in } \Omega \\
\psi|_{\partial \Omega} = 1
\end{array} \right. \quad (q = \frac{\nabla^2 \sqrt{\gamma}}{\sqrt{\gamma}}) \quad (1.95)
$$

where $u = \sqrt{\gamma}$ is the standard solution of the conductivity equation. We know that for each $k = k_1 + ik_2$, there exist the unique solution $\psi(\cdot, k)$ of

$$
-\nabla^2 \psi + q\psi = 0 \quad \text{in } \mathbb{R}^2, \quad e^{ik(x+iy)}(1 - \psi) \in W^{1,p}(\mathbb{R}^2)
$$

and the scattering transform of $q \in C_0(\Omega)$ can be expressed as

$$
t(k) = \int_{\mathbb{R}^2} e^{ik(x+iy)}q(x,y)\psi(x,y; k)dz \quad (z = (x,y), \quad k = k_1 + ik_2)
$$

$$
= \int_{\Omega} e^{ik(x+iy)}\nabla^2 \psi(z,k)dz
$$

$$
= \int_{\partial \Omega} e^{ik(x+iy)}(\Lambda_\gamma^D - \Lambda_1^D)\psi(z,k)ds \quad (\cdot \nabla^2 e^{ik(x+iy)} = 0)
where $\Lambda^D_\gamma : H^\frac{1}{2}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega)$ is the Dirichlet-to-Neumann(DtN) map given by $\Lambda^D_\gamma f = \gamma^D \left( \frac{\partial u^f}{\partial n} \right)_{|\partial \Omega}$ where $u^f$ is the solution of $\nabla \cdot (\gamma \nabla u^f) = 0$ in $\Omega$ with the Dirichlet boundary data $u^f|_{|\partial \Omega} = f$. Using the fact that $-\nabla^2 \psi + q \psi = 0$ and the above property of $t(x,k)$, it is easy to prove that

$$\mu(z,k) := e^{ik(x+iy)} \psi(z,k), \quad (k \notin \mathbb{C} \setminus \{0\})$$

satisfies the D-bar equation:

$$\frac{\partial}{\partial k} \mu(z,k) = \frac{t(k)}{4\pi k} e^{-2i(k_1 x - k_2 y)} \mu(z,k), \quad k \in \mathbb{C} \setminus \{0\} \quad (1.96)$$

From (1.95), solving the D-bar equation (1.96) for $\mu(z,k)$ leads to the reconstruction algorithm for $\gamma$:

$$\sqrt{\gamma(z)} = \lim_{k \to 0} \mu(z,k), \quad z = (x,y) \in \Omega$$

For the reconstruction algorithm, we need the following steps:
1. Compute $\psi(\cdot,k)|_{|\partial \Omega}$ for each $k = k_1 + ik_2$.
2. Compute $t(k)$ using step 1.
3. Solve the D-bar equation (1.96) for $\mu(z,k)$.
4. Visualize $\sqrt{\gamma(z)} = \lim_{k \to 0} \mu(z,k), \quad z = (x,y) \in \Omega$

For a precise explanation of the reconstruction algorithm, let us fix notations and definitions:
- For a complex variable $z = x + iy$ at a point $z = (x,y)$, define D-bar operator $\bar{\partial}$ by

  $$\bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

- For $k = k_1 + ik_2 \in \mathbb{C} \setminus \{0\}$,

  $$g_k(x,y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x_1+iy)\xi} \frac{e^{2ik_1x+2ik_2y}}{\sqrt{|\xi|^2 + 2k(\xi_1 + i\xi_2)}} d\xi$$

  Note that $g_k$ satisfies $(-\nabla^2 - 4ik\bar{\partial})g_k(x,y) = \delta(x,y)$.
- Define a single layer operator $S_k$ for $k = k_1 + ik_2 \in \mathbb{C} \setminus \{0\}$ by

  $$S_k \phi(z) = \int_{\partial \Omega} G_k(z-z')\phi(z')dz', \quad z = (x,y)$$

  where $G_k(z) = e^{ik(x+iy)}g_k(z)$. Note that $-\nabla^2 G_k(z) = \delta(z)$.

The direct method for reconstructing $\gamma$ without iteration is based on the following Nachmann’s constructive result.

**Theorem 1.12.7** 1. For each $k = k_1 + ik_2 \in \mathbb{C} \setminus \{0\}$, there exists a unique solution $\psi(\cdot,k) \in H^\frac{1}{2}(\partial \Omega)$ satisfying the integral equation

$$\psi(\cdot,k)|_{|\partial \Omega} = e^{ikz} - S_k(\Lambda^D_\gamma - \Lambda^D_\gamma)\psi(\cdot,k)$$

where $ikz = ik(x+iy)$ and $\Lambda^D_\gamma$ denotes DtN map of the homogeneous conductivity $\gamma = 1$. 


2. For each \( z = (x, y) \), the solution \( \mu \) of (1.96) satisfies the following integral equation
\[
\mu(z, k) = 1 + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{t(k')}{(k' - k)k'} e^{-2(k'_1x - k'_2y)} \mu(z, k') \, dk'_1 \, dk'_2
\]

3. The \( \gamma \) is reconstructed by
\[
\sqrt{\gamma}(z) = \lim_{k \to 0} \mu(z, k) \quad z = (x, y) \in \Omega
\]

**Proof.** For the detailed proof, please refer to (Nachmann 1996).

### 1.13 Back projection algorithm (Optional)

In 1983, Barber and Brown introduce the back projection algorithm for EIT, which was the first fast and useful algorithm in EIT although it provides images in very low resolution. Since this algorithm is motivated from the CT-algorithm, it can be viewed as a generalized Radon Transform. There exist a big difference between EIT and CT; In CT, we can obtain projected images with various direction, while in EIT we cannot control the pathway of currents since the flow of current itself depends on the distribution of conductivity to be imaged. Under assumption that the conductivity is small perturbation of a constant value, we can approximately apply the back projection algorithm.

Let us begin with reviewing the well-known Radon transform. In CT, we try to reconstruct a two dimensional cross sectional image \( f \) of a subject from its X-ray projections with several different directions \((\cos \theta, \sin \theta)\). The projection of \( f \) with \( \theta \)-direction can be defined by
\[
P_{\theta}f(t) = \int_{L_{\theta,t}} f \, dl \quad (L_{\theta,t} := \{(x, y) : x \cos \theta + y \sin \theta = t\})
\]

Taking Fourier transform of \( P_{\theta}f \) leads to
\[
\hat{P}_{\theta}f(k) = \int_{-\infty}^{\infty} P_{\theta}f(t)e^{-ikt} \, dt
\]
\[
= \int_{-\infty}^{\infty} \left[ \int_{L_{\theta,t}} f(x, y) \, dl \right] e^{-ikt} \, dt
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-ik(x \cos \theta + y \sin \theta)} \, dx \, dy
\]
\[
= \sqrt{2\pi} \hat{f}(k \cos \theta, k \sin \theta).
\]

The reconstruction algorithm is based on the following expression of \( f \) in terms of its projection:
\[
f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \hat{f}(k \cos \theta, k \sin \theta) e^{ik(x \cos \theta + y \sin \theta)} \, dk \right] \, d\theta
\]
\[
= \frac{1}{2\pi} \int_{0}^{\pi} \left[ \int_{-\infty}^{\infty} \hat{f}(k \cos \theta, k \sin \theta) e^{ik(x \cos \theta + y \sin \theta)} \, dk \right] \, d\theta
\]
\[
\approx \frac{1}{(2\pi)^{3/2}} \frac{\pi}{N} \sum_{j=1}^{N} \int_{-\infty}^{\infty} \left[ \hat{P}_{\theta_j}f(k) \right] e^{ik(x \cos \theta_j + y \sin \theta_j)} \, dk \]
where \( \theta_j = \frac{j\pi}{N} \). Hence, the image \( f(x,y) \) can be computed from the knowledge of its projection \( P_{\theta_j} f, \ j = 1, 2, \cdots, N \).

**Barber-Brown’s back projection algorithm**

To explain Barber-Brown’s back projection algorithm quickly, we assume the following:

- \( \Omega \) is the unit disk in \( \mathbb{R}^2 \).
- \( \gamma = \gamma_0 + \delta \gamma \) and \( \gamma_0 = 1 \).
- \( \delta \gamma \in C_0^2(\Omega) \).
- \( P_\theta = (\cos \theta, \sin \theta) \) and \( z = (x,y) \) (or \( z = x + iy \)).

Let \( u_0 \) and \( u \) denote the electric potentials corresponding \( \gamma_0 = 1 \) and \( \gamma \) with the same Neumann dipole boundary data \( g_{\theta} = \frac{2\pi}{\epsilon} (\delta P_{\theta} \cdot \nabla u_0) \).

Writing \( u = u_0 + \delta u \), \( \delta u \) satisfies approximately the equation

\[
-\nabla^2 \delta u \approx \nabla \delta \gamma \cdot \nabla u_0 \quad \text{in} \ \Omega
\]

\[
\frac{\partial \delta u}{\partial n} = 0 \quad \text{on} \ \partial \Omega \tag{1.97}
\]

Here, the term \( \nabla \delta \gamma \cdot \nabla \delta u \) was neglected. When \( \epsilon \) is very small, \( u_0 \) can be computed approximately as

\[
u_0(z) \approx \frac{z \cdot P_\theta^\perp}{|z - P_\theta|^2}, \quad z = (x,y)
\]

where \( P_\theta^\perp = (\cos(\theta + \pi/2), \sin(\theta + \pi/2)) \). Next, we introduce a holomorphic function in \( \Omega \) whose real part is \( -u_0 \):

\[
\Psi_\theta(z) = \xi + i\eta = -u_0(z) + iu_0(z) := -\frac{z \cdot P_\theta^\perp}{|z - P_\theta|^2} + i \frac{1 - P_\theta \cdot z}{|z - P_\theta|^2}.
\]

Then \( \Psi_\theta \) maps from the unit disk onto the upper half plane:

\[
\Psi_\theta : \Omega \rightarrow \Psi_\theta(\Omega) := \{ \xi + i\eta \mid \eta > \frac{1}{2} \}
\]

Define

\[
\widetilde{\delta u}(\xi + i\eta) := \delta u(\Psi_\theta^{-1}(\xi + i\eta)) \quad \text{and} \quad \widetilde{\delta \gamma}(\xi + i\eta) := \delta \gamma(\Psi_\theta^{-1}(\xi + i\eta))
\]

Viewing \( \xi = \xi(x,y) \) and \( \eta = \eta(x,y) \), we have

\[
\nabla \xi \cdot \nabla \eta = 0 \quad \& \quad |
\nabla \xi| = |
\nabla \eta|
\]

Hence, the perturbed equation (1.97) leads that \( \widetilde{\delta u} \) satisfies

\[
-\nabla^2 \widetilde{\delta u} = \frac{\partial \widetilde{\delta \gamma}}{\partial \xi} \quad \text{in} \ \Psi_\theta(\Omega), \quad \frac{\partial \widetilde{\delta u}}{\partial \eta} \bigg|_{\eta = \frac{1}{2}} = 0 \tag{1.98}
\]
For a moment, we assume that $\delta \gamma$ is independent of $\eta$-variable. With this temporary assumption, $\delta u$ is independent of $\eta$-variable and hence

$$\frac{\partial^2}{\partial \xi^2} \delta u = - \frac{\partial}{\partial \xi} \delta \gamma \quad \text{in } \Psi_\theta(\Omega)$$

Therefore, $\frac{\partial}{\partial \xi} \delta u = - \delta \gamma$ and

$$\delta \gamma(\xi, \eta) = - \frac{\partial \delta u}{\partial \xi}(\xi + \frac{1}{2})$$

For fixed $z$, denote $\Psi_\theta(z) = \xi_\theta + i \eta_\theta$ and $z_\theta^* = \Psi_\theta^{-1}(\xi_\theta + i \frac{1}{2})$. (see Figure ??.) Using the relation among $\Psi_\theta, z,$ and $z^*$, Barber and Brown derived the reconstruction formula

$$\delta \gamma(z) = \delta \gamma(\Psi_\theta(z)) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \xi} \delta u(\xi_\theta + i \frac{1}{2}) \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial T} \delta u(z_\theta^*) \, d\theta$$

(1.99)

where $\frac{\partial}{\partial T}$ denotes the tangential derivative at $z_\theta^* \in \partial \Omega$.

**Problems**

“There will be about 10 problems. This section will be completed later.”